[redacted]

1.2. Suppose that the spin magnitude is 1 on the even sites and $\mathcal{E}$ on the odd sites. Then the equations for the magnetization are:

Even sites: (magnitude 1 spins)

$$E(S_{even}) = -J S_{even} \sum S_j \to -J S_{even}\, 2d\, m_{odd}$$

$$\Rightarrow \boxed{m_{even} = \tanh 2dJ\beta m_{odd}}$$

Odd sites: $E(odd) = -J S_{odd} \sum S_j = -J S_{odd}\, 2d\, m_{even}$

$$\Rightarrow \boxed{m_{odd} = \mathcal{E} \tanh 2dJ\beta\mathcal{E} m_{even}}$$

For small m_{odd}, m_{even}, these become

$$\left.\begin{aligned} m_e &\simeq 2dJ\beta m_o \\ m_o &\simeq 2dJ\beta\mathcal{E}^2 m_e \end{aligned}\right\}$$

Eliminate m_o to find $kT_c = 2dJ\mathcal{E}$ for both even and odd sites.

[redacted]

Determine meven + modd :

Using $2B_c dJ\varepsilon = 1$, then $2BdJ = \frac{B}{\varepsilon B_c} = \frac{T_c}{\varepsilon T}$

Then $m_e = \tanh \frac{T_c}{\varepsilon T} m_o$; $m_o = \varepsilon \tanh \frac{T_c}{T} m_e$

For small m_o, m_e :

$$m_e \cong \frac{T_c}{\varepsilon T} m_o - \frac{1}{3}\left(\frac{T_c}{\varepsilon T}\right)^3 m_o^3 \quad ; \quad \frac{m_o}{\varepsilon} \cong \frac{T_c}{T} m_e - \frac{1}{3}\left(\frac{T_c}{T}\right)^3 m_e^3$$

To lowest order :

$$m_e \simeq \frac{T_c}{\varepsilon T}\left(\frac{\varepsilon T_c}{T} m_e - \frac{1}{3}\varepsilon\left(\frac{T_c}{T}\right)^3 m_e^3\right) - \frac{1}{3}\left(\frac{T_c}{\varepsilon T}\right)^3\left(\frac{\varepsilon T_c}{T}\right)^3 m_e^3$$

$$1 \simeq \left(\frac{T_c}{T}\right)^2 - \left(\frac{1}{3}\frac{T_c^4}{T^4} + \frac{1}{3}\frac{T_c^6}{T^6}\right) m_e^3$$

Near T_c this gives :

$$\frac{2}{3} m_e^2 \simeq \left[\left(\frac{T_c}{T}\right)^2 - 1\right] \simeq \left(\frac{T_c}{T} - 1\right)\left(\frac{T_c}{T} + 1\right) \simeq 2\left(\frac{T_c}{T} - 1\right)$$

$$m_e^2 \simeq 3\left(\frac{T_c}{T} - 1\right)$$

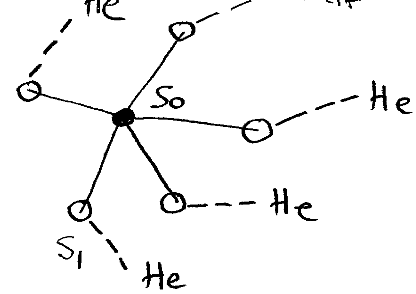
Similarly :

$$\frac{m_o}{\varepsilon} \simeq \frac{T_c}{T}\left(\frac{T_c}{\varepsilon T} m_o - \frac{1}{3}\left(\frac{T_c}{\varepsilon T}\right)^3 m_o^3\right) - \frac{1}{3}\left(\frac{T_c}{T}\right)^3\left(\frac{T_c}{\varepsilon T}\right)^3 m_o^3$$

$$1 \simeq \left(\frac{T_c}{T}\right)^2 - \left(\frac{1}{3}\frac{T_c^4}{T^4}\frac{1}{\varepsilon^2} + \frac{1}{3}\frac{T_c^6}{T^6}\frac{1}{\varepsilon^2}\right) m_o^2$$

$$\Rightarrow m_o^2 \simeq 3\varepsilon^2\left(\frac{T_c}{T} - 1\right)$$

1.2 Schematic of system in the Bethe approximation:



The z spins in the first shell experience an effective field He. They also interact with the central spin s_0

For this cluster, the energy is:

$$\mathcal{H}_{cluster} = -J s_0 \sum_{j\in 1} s_j - H_{eff} \sum_{j \in 1} s_j$$

$$Z_{cluster} = \sum_{\text{states}} e^{-\beta \mathcal{H}_{cl}} = \Big(\sum_{\substack{\text{states} \\ \text{with } s_0=+1}} + \sum_{\substack{\text{states} \\ \text{with } s_0=-1}} \Big) e^{-\beta \mathcal{H}_{cl}}$$

$$= \sum_{\{s_j\}} e^{+\beta(H_e+J)\sum_j s_j} + \sum_{\{s_j\}} e^{+\beta(H_e-J)\sum_j s_j}$$

$$= \sum_{\{s_j\}} \Big(\prod_j e^{\beta(H_e+J)s_j} + \prod_j e^{\beta(H_e-J)s_j} \Big)$$

but since each s_j is independent each factor in the product is equal

$$= \sum_{s_j} \left(e^{\beta(H_e+J)s_j}\right)^z + \left(e^{\beta(H_e-J)s_j}\right)^z$$

$$= [2\cosh\beta(H_e+J)]^z + [2\cosh\beta(H_e-J)]^z$$

Clearly $\langle s_0 \rangle = [2\cosh\beta(H_e+J)]^z - [2\cosh\beta(H_e-J)]^z / Z_{cl}$

For the mean value of the spin in the 1st shell, we can independently sum over the spin states of the other spins in the 1st shell. Therefore,

$$\langle s_i \rangle = \frac{1}{Z_{cl}} \Big\{ \underbrace{\underbrace{[2\cosh\beta(H_e+J)]^{z-1}}_{\text{sum over spins } j\neq i} \underbrace{2\sinh\beta(H_e+J)}_{\text{sum over spin } i}}_{\text{states with } s_0=+1} + \underbrace{[2\cosh\beta(H_e-J)]^{z-1} 2\sinh(H_e-J)}_{\text{states with } s_0=-1} \Big\}$$

The condition $\langle s_0 \rangle = \langle s_i \rangle$ gives:

$$\sinh A (\cosh A)^{z-1} + \sinh B (\cosh B)^{z-1} = (\cosh A)^z + (\cosh B)^z$$

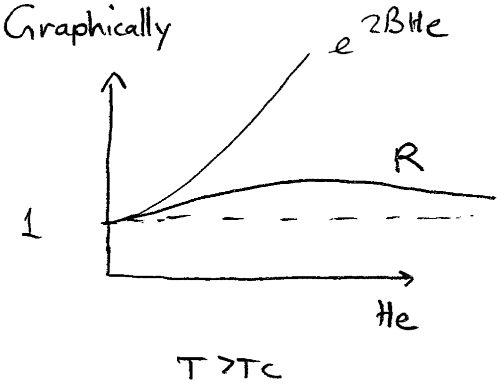
with $A = \beta(H_e+J)$; $B = \beta(H_e-J)$

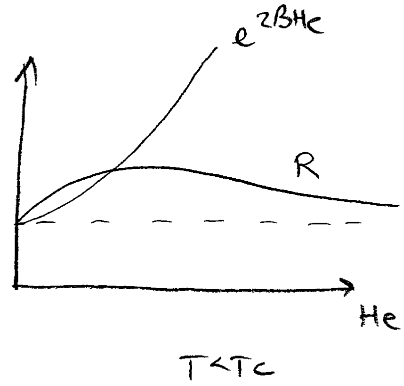
This can be simplified a bit by dividing by $(\cosh B)^{z-1}$ and defining $R = (\cosh A/\cosh B)^{z-1}$. This gives:

$$R \sinh A + \sinh B = R \cosh A + \cosh B$$

$$R = \frac{\cosh B - \sinh B}{\sinh A - \cosh A} \Rightarrow e^{2\beta H e} = \left(\frac{\cosh \beta (He+J)}{\cosh \beta (He-J)} \right)^{z-1} \qquad (1)$$

Graphically





To find the non-trivial solution, compute $\frac{\partial}{\partial H}$ of (1) at $H=0$

$$\frac{\partial}{\partial H} e^{\frac{2}{z-1}\beta H}\Big|_{H=0} = \frac{2\beta}{z-1} \quad ; \quad \frac{\partial}{\partial H} R^{\frac{1}{z-1}} = \beta \frac{\sinh \beta(H+J)}{\cosh \beta(H-J)} - \frac{\beta \cosh \beta(H+J) \sinh \beta(H-J)}{\cosh^2 \beta(H-J)}$$

$$= 2 \tanh \beta J \quad (\text{at } H=0)$$

$\Rightarrow z-1 = \coth \beta_c J$ with solution $\frac{kT_c}{J} \simeq 2.885$ for $z=4$ (Compared to $\frac{kT_c}{J} = 4$ in Curie Weiss theory, and $\frac{kT_c}{J} \cong 2.27$, exact)

3. For an N-site complete graph, when there are k spins ↓ and N-k spins ↑, the energy gap with respect to the ground state is $\Delta E = 2Jk(N-k)$

For $H=0$, $Z_N = e^{-\beta E_0} \sum_{k=0}^{N} \binom{N}{k} e^{-2\beta J k(N-k)}$

Use Stirling's approximation for the binomial, write $j = NJ$ and $x = k/N$. Then,

$$Z_N \sim e^{-\beta E_0} \int_0^1 N dx \, e^{N \underbrace{(-x \ln x - (1-x)\ln(1-x) - 2\beta j x(1-x))}_{\equiv f(x)}}$$

Evaluate integral by the Laplace method

$f(x) = -x\ln x - (1-x)\ln(1-x) - 2Bjx(1-x)$

$f'(x) = \ln\left(\frac{1-x}{x}\right) - 2Bj(1-2x)$

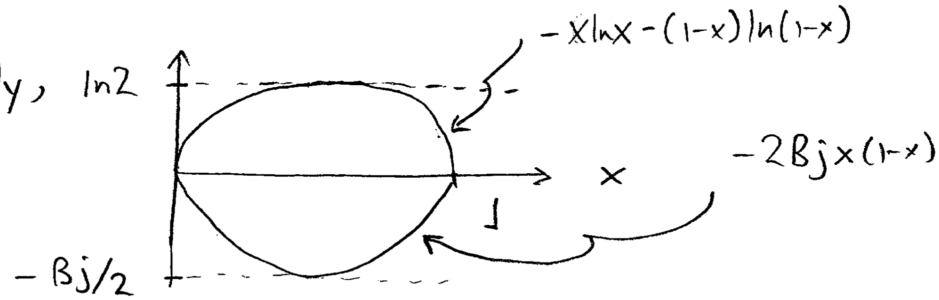
$f''(x) = -\frac{1}{x} - \frac{1}{1-x} + 4Bj$

By inspection $f'(x) = 0$ has a solution at $x = 1/2$. Then $f''(\frac{1}{2}) = 4(Bj-1)$. Hence for $Bj < 1$, $x = 1/2$ maximizes $f(x)$ and states near $x = 1/2$ dominate in the integral $\Rightarrow$ paramagnetic phase.

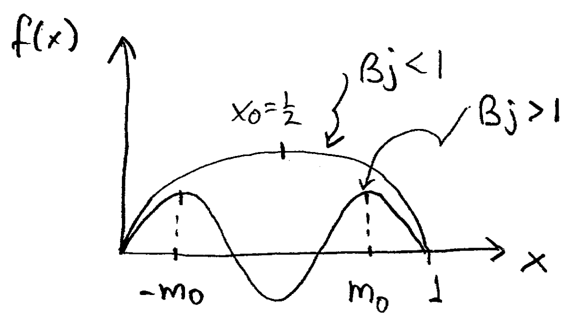
For $Bj > 1/2$, $x = 1/2$ minimizes $f(x)$ and other solutions to $f'(x)$ must exist. To find them, note that $1 - 2x = m$, the magnetization per spin. So $x = (1-m)/2$, $(1-x) = (1+m)/2$ and $f'(x) = 0$ becomes:

$$-2Bjm = \ln\left(\frac{1-m}{1+m}\right), \text{ or } \boxed{m = \tanh Bjm} \quad \text{the mean field solution!}$$

Graphically,



For $Bj < 1$, the positive contribution of $-x\ln x - (1-x)\ln(1-x)$ is larger than $-2Bjx(1-x)$ for all $0 \leq x \leq 1$ and vice versa. Therefore



$$Z \sim \int e^{Nf(x)}\,dx \; e^{-\beta E_0}$$

$$\sim e^{-\beta E_0 + N f(x_0)} \sqrt{\frac{2\pi}{f''(x_0)}} \times g$$

where $x_0 = 1/2$ in the paramagnetic phase and $g = 1$; and x_0 determined by the condition $m = \tanh Bjm$ and $g = 2$ (two peaks) in the ferromagnetic phase.