FINAL EXAM **PY541** **December 16, 2005**

No notes, books or calculators are allowed or needed. Please **explain your work clearly** to maximize the probability of receiving partial credit. Long calculations are unnecessary to solve the problems. Note also that later parts of a question do not always require the answer to early parts — be sure to attempt all parts of a question. If you are truly stuck, you may request a hint at the cost of an appropriate number of points.

1. (`9 points`) An ideal gas is confined within a vertical cylinder by a freely sliding piston of mass m. Both the cylinder and piston have a cross-sectional area A. In gravitational equilibrium, the volume of the gas is V_0. Under the assumption of an adiabatic process, determine the frequency of small oscillations of the piston when it is displaced slightly from equilibrium. Express your result in terms of m, g, A, V_0, p_0 (atmospheric pressure), and $\gamma \equiv C_p/C_V$.

   (For partial credit, solve this problem under the assumption of isothermal conditions.)

2. (`9 points`) In a simple metal, the minimum energy required to remove an electron at the top of the Fermi sea from the metal is the work functionΦ. Consider the thermal equilibrium of the free electron gas outside the metal with electrons in the metal at low temperatures, $kT \ll \Phi$.

   (a) As a preliminary, calculate the partition function of an ideal gas and thereby determine the chemical potential of the free electron gas outside the metal.

   (b) Equate the chemical potential of the electrons within and outside the metal to find the equilibrium density of electrons outside the metal. (Note: In (a) and (b) you need to express all energies consistently with respect to a common zero, as indicated in the diagram to the right.)

   (c) Estimate the electron density numerically at room temperature, under the assumption that $\Phi \cong 0.1$eV.

3. (`9 points`) Consider an ideal Bose gas a low temperatures. The number density of the gas is given by

$$n = \frac{g_{3/2}(z)}{\lambda^3} + \frac{1}{V}\frac{z}{1-z};$$

   a sketch of $g_{3/2}(z)$ is shown to the right.

(a) Determine the occupancy of the ground state for this ideal Bose gas. Express your result separately for (i) $T > T_c$ and (ii) $T < T_c$. Write the expression for T_c.

(b) For $T < T_c$, determine the temperature dependence of the mean energy $\langle U \rangle$ (a detailed calculation is not needed to obtain only the temperature dependence). Is the dependence of $\langle U \rangle$ on T sub-linear, linear, or super-linear? Give a physical explanation for your result.

4. (`11 points`) Consider a Van der Waals gas, with equation of state

$$(p + \frac{a}{v^2})(v - b) = kT.$$

In a microscopic picture for this gas, b is of the order of v_0, the volume occupied by each molecule, and a is of order $u_0 v_0$, where u_0 is the energy minimum in the interparticle potential. For a typical inert gas that can be modeled by the Van der Waals equation of state (such as argon) $u_0 \approx 10^{-2}$eV.

(a) At room temperature and pressure, estimate numerically the ratio b/v and a/pv^2 in the Van der Waals equation of state, using the above value of u_0 and by making an estimate for v_0. With these results, estimate the accuracy of the ideal gas law for this Van der Waals gas. (Note: Atmospheric pressure is approximately given by 1 atm $\approx 10^6$ erg/cm^3, and 1eV $\approx 1.6 \times 10^{-12}$erg.

(b) Sketch a subcritical, critical, and supercritical isotherm. State the two conditions on derivatives of $p(v)$ with respect to v that specify the critical point.

(c) Express a and b in terms of the critical temperature T_c and critical volume per molecule v_c. Determine the order of magnitude of T_c in terms of u_0 and v_0.

5. (`12 points`) Consider an Ising spin system with a nearest-neighbor ferromagnetic interaction J and zero magnetic field. Suppose that the spin up state is doubly degenerate, while the spin down state is non-degenerate.

(a) For 2 spins, determine the limiting magnetization per spin as (i) $T \to \infty$ and (ii) as $T \to 0$. Interpret your results physically.

(b) For an N-spin system with periodic boundary conditions, compute the partition function. (For partial credit, compute the partition function of a 2-spin system with periodic boundary conditions.)

(c) Determine the nearest-neighbor correlation function $\langle s_i s_{i+1} \rangle$. Determine the limiting value of this function as (i) $T \to \infty$ and (ii) $T \to 0$? Interpret your results physically.

(For partial credit, solve this problem under the assumption of both spin states being non-degenerate.)