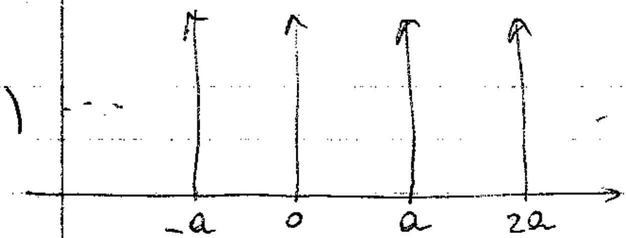
PY 452 Problem Set 2

1)



$k = \frac{\sqrt{2mE}}{\hbar}$

$K = \frac{2\pi n}{Na}$

$E > 0$ $\quad 0 < x < a \quad \psi_R(x) = A\sin(kx) + B\cos(kx)$

$-a < x < 0$ Bloch's th. $\psi_L(x) = e^{-iKa}\psi_R(x+a)$

at $x = 0$ $\quad \boxed{\psi_L(0) = \psi_R(0)}$

$$\boxed{\left.\frac{d\psi_R}{dx}\right|_{x=0} - \left.\frac{d\psi_L}{dx}\right|_{x=0} = -\frac{2m\alpha}{\hbar^2}\psi(0)}$$

$$\rightarrow \boxed{\cos(Ka) = \cos(ka) - \frac{m\alpha}{\hbar^2 k}\sin(ka)}$$

for $E < 0$ $\quad k \rightarrow ik$

$$\boxed{\cos(Ka) = \cosh ka + \frac{m\alpha}{\hbar^2 k}\sinh ka}$$

$\beta = \frac{m\alpha a}{\hbar^2} \quad \rightarrow f(z) = \cosh z + \beta\frac{\sinh z}{z}$

$\beta = -1,5$

allowed energies $\boxed{|f(z)| \leq 1}$

Eeach band contains N states.

2) a)

$$\psi(x_1, x_2, x_3) = \frac{1}{\sqrt{6}}\left(\frac{2}{a}\right)^{3/2} \sum_{i,j,k=1}^{3} \epsilon_{ijk} \sin\left(\frac{5\pi}{a} x_i\right) \sin\left(\frac{7\pi}{a} x_j\right) \sin\left(\frac{17\pi}{a} x_k\right)$$

where $\epsilon_{ijk} = \begin{cases} +1 & (i,j,k) = (1,2,3), (3,1,2) \text{ or } (2,3,1) \\ 0 & \text{if } i=j \text{ or } j=k \text{ or } i=k \\ -1 & (i,j,k) = (1,3,2), (3,2,1) \text{ or } (2,1,3) \end{cases}$

b) i)

$$\psi = \left(\sqrt{\frac{2}{a}}\right)^3 \sin\left(\frac{11\pi x_1}{a}\right) \sin\left(\frac{11\pi x_2}{a}\right) \sin\left(\frac{11\pi x_3}{a}\right)$$

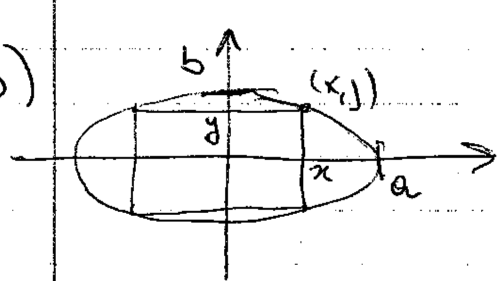
ii) $\psi = \frac{1}{\sqrt{3}}\left(\frac{2}{a}\right)^{3/2} \Big[\sin\left(\frac{\pi x_1}{a}\right) \sin\left(\frac{\pi x_2}{a}\right) \sin\left(\frac{19\pi}{a} x_3\right) +$

all possible permutations of $x_1, x_2, x_3 \Big]$

iii) $\psi = \frac{1}{\sqrt{6}}\left(\frac{2}{a}\right)^{3/2} \Big[\sin\left(\frac{5\pi x_1}{a}\right) \sin\left(\frac{7\pi}{a} x_2\right) \sin\left(\frac{17\pi}{a} x_3\right) +$

all possible permutations of x_1, x_2 and $x_3 \Big]$

3)



$$A(x,y) = 4xy$$

$$\left(x/a\right)^2 + \left(y/b\right)^2 = 1$$

$$G = A + \lambda\left[\left(x/a\right)^2 + \left(y/b\right)^2 - 1\right]$$

$$\frac{\partial G}{\partial x} = \frac{\partial G}{\partial y} = \frac{\partial G}{\partial \lambda} = 0$$

$$\hookrightarrow \quad y = -\frac{\lambda x}{2a^2} \longrightarrow x = \frac{\lambda^2}{4a^2b^2}\,x \quad \begin{cases} x = 0 \\ \text{or } \boxed{\lambda = \pm 2ab} \end{cases}$$

$y = \mp \frac{b}{a} x$ choosing $y > 0$

$$\frac{x^2}{a^2} + \frac{b^2}{a^2}\frac{x^2}{b^2} = 1 \rightarrow x = \frac{a}{\sqrt{2}} \rightarrow \boxed{A = 2ab}$$

4)

$$E = \frac{\hbar^2 k^2}{2m}, \quad \vec{k} = \left(\frac{2\pi n_x}{l_x}, \frac{2\pi n_y}{l_y}\right) \quad dk_x = \frac{2\pi}{l_x}; \; dk_y = \frac{2\pi}{l_y}$$

$$\rightarrow \boxed{d^2k = \frac{4\pi^2 d^2n}{A}}$$

two per state

$$N_q = 2\int d^2n = \frac{2A}{4\pi^2}\int d^2k = \frac{2A}{4\pi}\int_0^{k_F} k\,dk \int_0^{2\pi} d\theta = \frac{A k_F^2}{2}$$

$$k_F^2 = \pi\frac{2N_q}{A} = 2\sigma\pi \longrightarrow \boxed{E_F = \frac{\hbar^2 k_F^2}{2m} = \frac{\pi\hbar^2\sigma}{m}}$$

in d-dim $\quad d^dk = \frac{(2\pi)^d}{V} d^dn$

$$\boxed{N_q} = 2\int d^dn = \frac{2V}{(2\pi)^d}\int d^dk = \boxed{\frac{2V}{(2\pi)^d}\frac{\pi^{d/2} k_F^d}{\Gamma(d/2+1)}}$$

gamma fn.

$$E_F = \frac{\hbar^2 k_F^2}{2m} = \frac{\hbar^2 \pi}{2m}\left[2^{d-1}\rho\,\Gamma(d/2+1)\right]^{2/d} \quad ; \quad \rho = \frac{N_q}{V}$$

5) a) $$\boxed{E = \frac{2\hbar^2}{15\pi m R^2}\left(\frac{9}{4}\pi N_q\right)^{5/3}}$$

b) $$\boxed{E_{grav} = -\frac{3}{5} G\frac{N^2M^2}{R}}$$

c) $E_{tot} = \frac{\alpha}{R^2} - \frac{\beta}{R} \qquad \frac{dE_{tot}}{dR} = 0 \rightarrow \boxed{R_{min} = \left(\frac{9\pi}{4}\right)^{2/3}\frac{\hbar^2}{GmM^2}\frac{q^{5/3}}{N^{1/3}}}$

d) $R = 7{,}16\ 10^6\ m$

e) $E_F = \frac{\hbar^2}{2mR^2}\left(\frac{9\pi}{4}N_q\right)^{2/3} = 1{,}94\ 10^5\ eV$