1) $V(r) = 0 \qquad r < a$

$V(r) = \infty \qquad r > a$

$$\psi(\vec{r}) = \Theta(\theta)\, R(r)$$

Free Particle Hamiltonian

$$\left(\frac{\vec{p}^{\,2}}{2m} - E\right)\psi = 0 \;\rightarrow\; \left(-\frac{\hbar^2}{2m}\nabla^2 - E\right)\psi = 0$$

$$\underbrace{-\frac{\hbar^2}{2m}}_{\alpha}\left[\frac{1}{r}\frac{\partial}{\partial r}\left(r\frac{\partial}{\partial r}\right) + \frac{1}{r^2}\frac{\partial^2}{\partial\theta^2} - \frac{E}{\alpha}\right]\Theta R = 0$$

$$\cancel{\alpha}\left[\frac{\partial^2 R}{\partial r^2} + \frac{1}{r}\frac{\partial R}{\partial r}\right]\Theta + \frac{\cancel{\alpha} R}{r^2}\frac{\partial^2\Theta}{\partial\theta^2} - \cancel{\alpha}\frac{E}{\alpha}\Theta R = 0 \qquad \div \frac{\Theta R}{r^2}$$

$$\underbrace{\frac{1\cancel{\alpha}}{R}\left[r^2\frac{\partial^2 R}{\partial r^2} + r\frac{\partial R}{\partial r}\right] - \cancel{\alpha}\frac{r^2 E}{\alpha}}_{= K^2} + \underbrace{\frac{\partial^2\Theta}{\partial\theta^2}\cdot\frac{1}{\Theta}}_{= -K^2} = 0$$

$$\rightarrow \quad \Theta(\theta) = A e^{ik\theta} + B e^{-ik\theta}$$

$$R(r) = J_0(Kr)$$

② $\langle E \rangle = \left(\frac{4}{6}\right)^2 E_1 + \left(\frac{3}{6}\right)^2 E_2 + \left(\frac{-1}{6}\right)^2 E_2 + \left(\frac{\sqrt{10}}{6}\right)^2 E_2$

$$= \frac{-mc^2\alpha^2}{2} \cdot \frac{21}{36}$$

$$\langle L^2 \rangle = \hbar^2 \left[\left(\frac{4}{6}\right)^2 \times 0 + \frac{20}{6^2} \times 2 \right] = \frac{10}{9}\hbar^2$$

$$\langle L_z \rangle = \hbar \left[\left(\frac{4}{6}\right)^2 \times 0 + \left(\frac{3}{6}\right)^2 \times 1 + \left(\frac{-1}{6}\right)^2 \times 0 + \left(\frac{\sqrt{10}}{6}\right)^2 \times (-1) \right] = -\frac{\hbar}{36}$$

sum over repeated index

③ $[H, \vec{r}.\vec{p}] = \left[\frac{p_i p_i}{2m} + V(r), r_j p_j \right] = \frac{1}{2m}[p_i p_i, r_j p_j] + [V(r), r_j p_j]$

$$= \frac{\cancel{2} p_i}{\cancel{2} m} \underbrace{[p_i, r_j]}_{-i\hbar\delta_{ij}} p_j + r_j \underbrace{[V(r), p_j]}_{= i\hbar \frac{\partial V(r)}{\partial x_j}} = -i\hbar \left(\frac{p_i p_i}{m} - r_j \frac{\partial V(r)}{\partial r_j} \right)$$

$$\rightarrow \langle [H, \vec{r}.\vec{p}] \rangle = 0 \rightarrow \left\langle \frac{\vec{p}^2}{m} \right\rangle = \langle \vec{r}. \nabla V(r) \rangle$$

$$V(r) = -\frac{Ze^2}{r} \rightarrow \vec{r}.\nabla V = \frac{Ze^2}{r} = -V(r) \Rightarrow \boxed{\left\langle \frac{\vec{p}^2}{2m} \right\rangle = \left\langle \frac{Ze^2}{2r} \right\rangle = -\frac{1}{2}\langle V(r) \rangle}$$

$$\langle E \rangle = \left\langle \frac{p^2}{2m} \right\rangle - \underbrace{\left\langle \frac{Ze^2}{r} \right\rangle}_{= \left\langle \frac{p^2}{m} \right\rangle} = -\left\langle \frac{p^2}{m} \right\rangle = \frac{-13{,}6\,eV}{n^2} = -\frac{R_y}{n^2}$$

$$\boxed{\left\langle \frac{1}{r} \right\rangle_{n,l} = \frac{1}{Ze^2} \left\langle \frac{p^2}{m} \right\rangle = \frac{R_y}{Ze^2 n^2}} \qquad \boxed{\langle T \rangle = \left\langle \frac{p^2}{2m} \right\rangle = \frac{R_y}{2n^2}}$$