① Writting $x=\sqrt{\frac{\hbar}{2m\omega}}\,(a+a^\dagger)$ $a|n\rangle=\sqrt{n}\,|n-1\rangle$

$[a,a^\dagger]=1$

$p=i\sqrt{\frac{m\omega\hbar}{2}}\,(a^\dagger-a)$

$$\langle m|x|n\rangle=\sqrt{\frac{\hbar}{2m\omega}}\Big(\underbrace{\langle m|a|n\rangle}_{=\,\sqrt{n}\langle m|n-1\rangle}+\langle m|a^\dagger|n\rangle\Big)=\sqrt{\frac{\hbar}{2m\omega}}\left(\sqrt{n}\,\delta_{m,n-1}+\sqrt{m-1}\,\delta_{m-1,n}\right)$$

$$\sqrt{n}\langle m|n-1\rangle+\sqrt{m-1}\langle m-1|n\rangle$$

$$\langle m|p|n\rangle=i\sqrt{\frac{m\hbar\omega}{2}}\Big(\langle m|a^\dagger|n\rangle-\langle m|a|n\rangle\Big)=i\sqrt{\frac{m\omega\hbar}{2}}\left(\sqrt{n+1}\,\delta_{m,n+1}-\sqrt{n}\,\delta_{m,n-1}\right)$$

② $\langle n|x^2|n\rangle=\frac{\hbar}{2m\omega}(2n+1)$ using the expressions of x and p in terms of a, $a^\dagger$

$\langle n|p^2|n\rangle=\frac{m\omega\hbar}{2}(2n+1)$

$$x^2=\frac{\hbar}{2m\omega}(a+a^\dagger)(a+a^\dagger)=\frac{\hbar}{2m\omega}\Big(a^2+2\overbrace{a^\dagger a}^{=N}+a^{\dagger 2}+1\Big)$$

$$\rightarrow\langle n|x^2|n\rangle=\frac{\hbar}{2m\omega}\Big(\underbrace{\cancel{\langle n|a^2|n\rangle}^{\,0}}_{\propto|n-2\rangle}+2\overbrace{\langle n|N|n\rangle}^{n}+\underbrace{\cancel{\langle n|a^{\dagger 2}|n\rangle}^{\,0}}_{\propto|n+2\rangle}+1\Big)$$

$$\boxed{\langle n|x^2|n\rangle=\frac{\hbar}{2m\omega}(2n+1)}$$

similarly for p^2.

$K=\frac{p^2}{2m}=V=m\omega^2\frac{x^2}{2}=\frac{\hbar\omega}{2}(n+1/2)=\frac{E}{2}$ just like in classical mechanics.

③ (6-58) $\sqrt{n!}\left(\frac{\hbar\pi}{m\omega}\right)^{1/4} u_n(x) = \left(\alpha x - \frac{1}{2\alpha}\frac{d}{dx}\right)^n e^{-\alpha^2 x^2}$

with $\alpha = \sqrt{\frac{m\omega}{2\hbar}}$ def $y = \alpha x$

$$\sqrt{n!}\left(\frac{\hbar\pi}{m\omega}\right)^{1/4} u_n(y) = \left(y - \frac{1}{2}\frac{d}{dy}\right)^n e^{-y^2} = v_n(y)$$

$$v_1(y) = \left(y - \frac{1}{2}\frac{d}{dy}\right)e^{-y^2} = 2y\,e^{-y^2}$$

$$v_2(y) = \left(y - \frac{1}{2}\frac{d}{dy}\right)\left(2y e^{-y^2}\right) = (4y^2 - 1)\,e^{-y^2}$$

$$v_3(y) = \left(y - \frac{1}{2}\frac{d}{dy}\right)\left(4y^2 - 1\right)e^{-y^2} = \left(8y^3 - 6y\right)e^{-y^2}$$