Solns Problem Set 6

1) Beginning with Eqns 4-25

$$\frac{igAe^{-iga} - igBe^{iga}}{Ae^{-iga} + Be^{iga}} = \frac{ike^{-ika} - ikRe^{ika}}{e^{-ika} + Re^{ika}} \qquad (a)$$

$$\frac{igAe^{iga} - igBe^{-iga}}{Ae^{iga} + Be^{-iga}} = \frac{ikTe^{ika}}{Te^{ika}} = ik \qquad (b)$$

LET $\phi = iga \quad \theta = ik \quad$ w/ (b)

$$\frac{B}{A} = \frac{(\phi - \theta)}{(\phi + \theta)} e^{2\phi}$$

from (a) $\Rightarrow$ $$\frac{\phi e^{-\phi} - \phi \frac{B}{A} e^{\phi}}{e^{-\phi} + \frac{B}{A} e^{\phi}} = \frac{\theta e^{-\theta} - \theta R e^{\theta}}{e^{-\theta} + Re^{\theta}}$$

$$y \equiv Re^{2\theta} \qquad z \equiv \frac{B}{A} e^{\phi}$$

$$X \equiv \phi\left(\frac{1-z}{1+z}\right) = \frac{\theta(1-y)}{(1+y)}$$

With these substitutions the soln is found

$$R = \frac{(\theta^2 - \phi^2)(1 - e^{4\phi})}{\theta\phi(1 + e^{4\phi}) + (\theta^2 + \phi^2)(1 - e^{4\phi})}$$

a) $J = \frac{\hbar}{2mi}\left(\psi \frac{d\psi^*}{dx} - \psi^* \frac{d\psi}{dx}\right)$

~~###~~ subst. $J_L = \frac{\hbar k}{m}(|A|^2 - |B|^2)$, $J_R = \frac{\hbar k}{m}(|C|^2 - |D|^2)$

so $|A|^2 + |D|^2 = |B|^2 + |C|^2$

using $C = S_{11} A + S_{12} D$
$B = S_{21} A + S_{22} D$

$\to$ $|A|^2 + |D|^2 = (S_{21}A + S_{22}D)(S_{21}^* A^* + S_{22}^* D^*) + (S_{11}A + S_{12}D)(S_{11}^* A^* + S_{12}^* D^*)$

since A and D are arbitrary

$|S_{21}|^2 + |S_{11}|^2 = |S_{22}|^2 + |S_{12}|^2 = 1$ and $S_{12} S_{22}^* + S_{11} S_{12}^* = 0$

To prove S is unitary just calculate SS^+ and S^+S using the above equalities.

b) i) $\phi(x) = \begin{cases} u_L(x) = Ae^{ikx} + Be^{-ikx} & x<0 \\ u_R(x) = Ce^{ikx} + De^{-ikx} & x>0 \end{cases}$ $k = \sqrt{\frac{2mE}{\hbar^2}}$

at $x=0$ $u_L(0) = u_R(0) \to \boxed{A+B = C+D}$

and integrating Sch. eqn between 0^+ and 0^-

$\phi'(0+) - \phi'(0-) = -\frac{2mV_0}{\hbar^2}\phi(0)$

$\boxed{ik(C-D-A+B) = -\frac{2mV_0}{\hbar^2}(C+D)}$ $\to$ $C = \frac{ik}{ik - \frac{mV_0}{\hbar^2}} A - \frac{mV_0/\hbar^2}{ik - mV_0/\hbar^2} D$

$\boxed{B = \frac{-mV_0/\hbar^2}{ik - mV_0/\hbar^2} A + \frac{ik}{ik - \frac{mV_0}{\hbar^2}} D}$

$S = \begin{pmatrix} t & r \\ r & t \end{pmatrix}$

ii) Using that $S = \begin{pmatrix} t & r \\ r & t \end{pmatrix}$ from problem 1

we know

$$t = \frac{2qk\, e^{-2ika}}{2kq\cos(2qa) - i(q^2+k^2)\sin(2qa)}$$

and

$$r = \frac{i\, e^{-2ika}\,(q^2-k^2)\sin(2qa)}{2kq\cos(2qa) - i(q^2+k^2)\sin(2qa)}$$

3) The tunneling factor is given by

$$e^{S} \quad w/ \quad S = \int_{R}^{2R} dx \sqrt{\frac{2m(V(x)-E)}{\hbar^2}}$$

From the energy of the particle it is possible to find the time required to interact w/ the boundry

$$E = \frac{1}{2}mv^2 \quad \Rightarrow \quad t = \sqrt{\frac{2mR^2}{E}}$$

$$\text{Lifetime} = (\text{time for 1 interaction})\left(\frac{\text{Prob of tunneling}}{\text{interaction}}\right)$$

$$= \sqrt{\frac{2mR^2}{E}}\; e^{2S}$$

for the case of the step potential

$$t \approx 5\times10^{-16}\,s$$

for the case of the

$$t \approx 1\times10^{-16}\,s$$