Soln's to HW 10

1)

a) $\psi = \frac{1}{\sqrt{\pi a^3}} e^{-r/a}$ thus $\langle r^n \rangle = \frac{1}{\pi a^3} \int r^n e^{-2r/a} d^3\vec{r}$

$$= \frac{4\pi}{\pi a^3} \int_0^\infty r^{n+2} e^{-2r/a} dr$$

$$\langle r \rangle = \frac{4}{a^3} \int_0^\infty r^3 e^{-2r/a} dr = \frac{4}{a^3} 3! \left(\frac{a}{2}\right)^4 = \frac{3}{2} a$$

$$\langle r^2 \rangle = \frac{4}{a^3} \int_0^\infty r^4 e^{-2r/a} dr = \frac{4}{a^3} 4! \left(\frac{a}{2}\right)^5 = 3a^2$$

b) $\langle x \rangle = 0$ $\quad \langle x^2 \rangle = \frac{1}{3} \langle r^2 \rangle = a^3$

c) $\psi_{211} = R_{21} Y_1^1 = -\frac{1}{\sqrt{\pi a}} \frac{1}{8a^2} r e^{-r/2a} \sin\theta e^{i\phi}$

$$\langle x^2 \rangle = \frac{1}{\pi a} \frac{1}{(8a^2)^2} \int (r^2 e^{-r/a} \sin^2\theta)(r^2 \sin^2\theta \cos^2\phi) r^2 \sin\theta \, dr \, d\theta \, d\phi$$

$$= \frac{1}{64\pi a^5} \int_0^\infty r^6 e^{-r/a} dr \int_0^\pi \sin^5\theta \, d\theta \int_0^{2\pi} \cos^2\phi \, d\phi$$

$$= \frac{1}{64\pi a^5} (6! \, a^7) \left(2 \frac{2 \cdot 4}{1 \cdot 3 \cdot 5}\right) \left(\frac{1}{2} 2\pi\right) = 12a^2$$

2) $\psi = \frac{1}{\sqrt{\pi a^3}} e^{-r/a}$ $\qquad P = |\psi|^2 4\pi r^2 dr$

$P = \frac{4}{a^3} e^{-2r/a} r^2 dr = p(r) dr$ $\qquad p(r) = \frac{4}{a^3} r^2 e^{-2r/a}$

$$\frac{dp}{dr} = \frac{4}{a^3}\left[2re^{-2r/a} + r^2\left(-\frac{2}{a} e^{-2r/a}\right)\right]$$

$$\frac{8r}{a^3} e^{-2r/a}\left(1 - \frac{r}{a}\right) = 0 \quad \Rightarrow \boxed{r = a}$$

3) $V = -G\frac{Mm}{r}$ thus $\frac{e^2}{4\pi\varepsilon_0} \to GMm$

b) $a_0 = \left(\frac{4\pi\varepsilon_0}{e^2}\right)\frac{\hbar^2}{m}$ thus $a_g = \frac{\hbar^2}{GMm^2}$

$a_g = 2.34 \times 10^{-138}$ m

c) $E_n = -\left[\frac{m}{2\hbar^2}(GMm)^2\right]\frac{1}{n^2}$ $\qquad E_c = \frac{1}{2}V = -\frac{GMm}{2r_0}$

$$\frac{GMm}{2r_0} = \frac{m(GMm)^2}{2\hbar^2}\frac{1}{n^2}$$

$$n^2 = \frac{GMm^2}{\hbar^2} r_0 = \frac{r_0}{a_0} \qquad n = \sqrt{\frac{r_0}{a_0}}$$

$$n = \sqrt{\frac{1.496 \times 10^{11}}{2.34 \times 10^{-138}}} = 2.53 \times 10^{74}$$

4) $\psi_{1s}(r) = \left(\frac{\beta}{\sqrt{\pi}}\right)^{3/2} e^{-\beta^2 r^2/2}$

Thus the Prob is the square of the Int:

$$\int d^3r \left(\frac{\beta}{\sqrt{\pi}}\right)^{3/2} e^{-\frac{\beta^2 r^2}{2}} \frac{2}{\sqrt{4\pi}} \left(\frac{Z}{a_0}\right)^{3/2} e^{-\frac{Zr}{a_0}}$$

$$= \frac{4}{\pi^{1/4}} \left(\frac{Z\beta}{a_0}\right)^{3/2} \int_0^\infty r^2 dr\, e^{-\frac{\beta^2 r^2}{2}} e^{-\frac{Zr}{a_0}}$$

$$= \frac{4}{\pi^{1/4}} \left(\frac{Z\beta}{a_0}\right)^{3/2} \left(-2\frac{d}{d\beta^2}\right) \int_0^\infty dr\, e^{-\frac{\beta^2 r^2}{2} - \frac{Zr}{a_0}}$$

In the limits $a_0\beta$ go to 0 and ∞