PY 451: Quantum Physics I   Problem Set 6

Due date: Friday, February 27, 2009, by 5:00pm

1. Complete the steps to derive Eqs. (4-26) from Eqs. (4-25) in the text.

2. (adapted from the text 4-1 and Griffiths 2.52) Consider an arbitrary localized potential in the region $|x| \leq a$. For $x > |a|$, the potential is zero. Assume that the wave functions to the left and to the right of the potential have the respective forms:

$$\begin{aligned} u_L(x) &= Ae^{ikx} + Be^{-ikx} \\ u_R(x) &= Ce^{ikx} + De^{-ikx} \; . \end{aligned}$$

We can relate outgoing and ingoing amplitudes by the *scattering matrix* $\mathbf{S}$, defined by:

$$\begin{pmatrix} C \\ B \end{pmatrix} = \begin{pmatrix} S_{11} & S_{12} \\ S_{21} & S_{22} \end{pmatrix} \begin{pmatrix} A \\ D \end{pmatrix} \equiv \mathbf{S} \begin{pmatrix} A \\ D \end{pmatrix} \; .$$

(a) Using the conservation of the total flux, show that the following relations hold:

$$\begin{aligned} |S_{11}|^2 + |S_{21}|^2 &= 1 \\ |S_{12}|^2 + |S_{22}|^2 &= 1 \\ S_{11}S_{12}^* + S_{21}S_{22}^* &= 0. \end{aligned}$$

Use these to show that $\mathbf{S}$ is unitary; that is, $\mathbf{S}\mathbf{S}^\dagger = \mathbf{S}^\dagger\mathbf{S} = \mathbf{I}$, where $\mathbf{I}$ is the unit matrix, and $\mathbf{S}^\dagger$ denotes the Hermitian conjugate. The Hermitian congugate is obtained by taking the transpose of $\mathbf{S}$ and then the complex conjugate of each matrix element.

(b) Calculate the elements of the scattering matrix for: (i) the delta potential well, $V(x) = -V_0\,\delta(x)$; and (ii) the square-well potential well, $V(x) = -V_0$ for $|x| < a$, and $V(x) = 0$ for $|x| > a$.

3. (adapted from text 4-8). Consider the potentials shown in the figure below. In part (b), the potential for $R < x < 2R$ is $V(x) = V\left[1 - (x-R)^2/R^2\right]$. For both cases, calculate the lifetime of a particle of energy E that is initially within the well $0 < x < R$. In both cases, estimate this lifetime numerically for an electron, with $R = 1\,\text{Å}$, $V = 2\,\text{eV}$, and $E = 1\,\text{eV}$.