Quantum Mechanics I

Final Exam

Date: Thursday, May 8, 2008, 9:00am - 11:00am

This is a CLOSED book exam. There are 3 questions. Write your name in all pages.
Use the space in the back of each page if you need additional space to write your solutions.

1. An electron in a hydrogen atom is in the state described at $t = 0$ by the wavefunction

$$\Psi(t=0,\vec{r}) = \frac{1}{\sqrt{2}}\left[\psi_{100}(\vec{r}) + \psi_{210}(\vec{r})\right],$$

   where $\psi_{nlm}(\vec{r})$ is the eigenfunction with quantum numbers n (principal), l (total angular momentum), and m (z-component of angular momentum).

   (a) What is the expectation value of the energy in state $\Psi(t=0,\vec{r})$?

   (b) What is the expectation value of L^2 in state $\Psi(t=0,\vec{r})$?

   (c) Write down the wavefunction $\Psi(t,\vec{r})$ for the system at a time $t > 0$.

   (d) Compute the expectation values $\langle\Psi(t)|\, x \,|\Psi(t)\rangle$, $\langle\Psi(t)|\, y \,|\Psi(t)\rangle$, and $\langle\Psi(t)|\, z \,|\Psi(t)\rangle$ for the state at time t. [*Hint*: by exploring symmetries, many of the needed integrals vanish.]

   (e) What is the frequency of oscillation of $\langle\Psi(t)|z|\Psi(t)\rangle$?

   Useful formulae:

$$Y_0^0(\theta,\varphi) = \sqrt{\frac{1}{4\pi}} \qquad Y_1^0(\theta,\varphi) = \sqrt{\frac{3}{4\pi}}\cos\theta$$

$$R_{10}(r) = 2\,a^{-3/2}\,e^{-r/a} \qquad R_{21}(r) = \frac{1}{\sqrt{24}}\,a^{-3/2}\,\frac{r}{a}\,e^{-r/2a}$$

2. Consider an electron of mass m in a quantum box with a confining potential in the x, y, z-directions:

$$V(x,y,z) = \frac{1}{2} m\omega^2 \left(x^2 + y^2 + z^2\right).$$

This confining potential corresponds to three harmonic oscillators, one in each of the x, y and z directions, and the eigenstates $|n_x, n_y, n_z\rangle$ are labeled by three quantum numbers n_x, n_y, n_z.

(a) What are the values of the energies of the lowest three levels (the ground state, the first and the second excited states)? What are the degeneracies of these three levels (*i.e.* number of such states with the same energy)?

For each of the three spatial directions of the harmonic oscillator one defines raising and lowering operators:

$$a = \frac{1}{\sqrt{2}}\left(\sqrt{\frac{m\omega}{\hbar}}\,\hat{x} + i\sqrt{\frac{1}{m\hbar\omega}}\,\hat{p}_x\right),\; b = \frac{1}{\sqrt{2}}\left(\sqrt{\frac{m\omega}{\hbar}}\,\hat{y} + i\sqrt{\frac{1}{m\hbar\omega}}\,\hat{p}_y\right),\; c = \frac{1}{\sqrt{2}}\left(\sqrt{\frac{m\omega}{\hbar}}\,\hat{z} + i\sqrt{\frac{1}{m\hbar\omega}}\,\hat{p}_z\right)$$

$$a^\dagger = \frac{1}{\sqrt{2}}\left(\sqrt{\frac{m\omega}{\hbar}}\,\hat{x} - i\sqrt{\frac{1}{m\hbar\omega}}\,\hat{p}_x\right),\; b^\dagger = \frac{1}{\sqrt{2}}\left(\sqrt{\frac{m\omega}{\hbar}}\,\hat{y} - i\sqrt{\frac{1}{m\hbar\omega}}\,\hat{p}_y\right),\; c^\dagger = \frac{1}{\sqrt{2}}\left(\sqrt{\frac{m\omega}{\hbar}}\,\hat{z} - i\sqrt{\frac{1}{m\hbar\omega}}\,\hat{p}_z\right)`$$

(b) Write down the expression for the angular momentum component $L_z = \hat{x}\hat{p}_y - \hat{y}\hat{p}_x$ in terms of the raising and lowering operators above.

(c) Suppose one makes a superposition

$$|\Psi(t=0)\rangle = \frac{1}{\sqrt{2}}\,|1,0,1\rangle + \frac{1}{2}\,|0,1,1\rangle + \frac{1}{2}\,|0,0,2\rangle$$

Compute $\langle\Psi(t)|L_z|\Psi(t)\rangle$ at a time $t > 0$.

(d) Write the Hamiltonian in terms of the raising and lowering operators, and compute its commutator with the operator L_z.

3. A coherent state of the harmonic oscillator, with Hamiltonian $\hat{H} = \hbar\omega\,(a^\dagger a + 1/2)$, is defined as the state

$$|\lambda\rangle = C(\lambda)\; e^{\lambda a^\dagger}|0\rangle$$

where $a, a^\dagger$ are the lowering and raising operators, $|0\rangle$ is the ground state of the harmonic oscillator, and $\lambda \in \mathbb{C}$ is a parameter characterizing the coherent state. $C(\lambda)$ is a normalization factor so that $\langle\lambda|\lambda\rangle = 1$.

(a) The coherent state above can be expanded in the basis of energy eigenstates $|n\rangle$, *i.e.*,

$$|\lambda\rangle = \sum_{n=0}^{\infty} c_n |n\rangle \ .$$

Calculate the expansion coefficients c_n.

(b) Write down the expression for the norm of the coherent state, *i.e.*, $\langle\lambda|\lambda\rangle$, in terms of the coefficients c_n. Determine the normalization constant $C(\lambda)$ such that $\langle\lambda|\lambda\rangle = 1$.

(c) Calculate the expected value of energy $E(\lambda) = \langle\lambda|\hat{H}|\lambda\rangle$ as a function of λ.