Quantum Mechanics I

Exam II

Date: Tuesday, March 25, 2008, 9:30am - 11:00am

This is a CLOSED book exam. There are 3 questions. Write your name in all pages.
Use the space in the back of each page if you need additional space to write your solutions.

1. Consider a square potential barrier $V(x)$:

$$V(x) = \begin{cases} 0 & \text{if } x < -a, \\ V > 0 & \text{if } -a < x < a \\ 0 & \text{if } x > a. \end{cases}$$

A particle with energy $E > V$ is incident from the left.

(a) Write down the form of generic solutions of Schrödinger's equation for $x < 0$ and $x > 0$. Express your answer in terms of wavenumbers k, k' on the two sides, and write down the relation between k, k' and E, V (and also $\hbar, m$).

(b) Obtain the boundary conditions (BCs) that the wavefunctions must obey at $x = \pm a$.

(c) From the relation between k, k' and E, V (and also $\hbar, m$) in part (a), find the energy E for which $k'a = \pi$.

(d) Solve for the amplitudes in part (a) using the BCs in part (b) for the specific case in part (c), *i.e.*, when $k'a = \pi$ **only**.

(e) Calculate the transmission and reflection coefficients for the specific energy in part (c).

2. Consider a potential wall at the origin and a delta function at $x = a > 0$:

$$V(x) = V_B(x) + V_0\,\delta(x - a)$$

where

$$V_B(x) = \begin{cases} \infty & \text{if } x < 0, \\ 0 & \text{if } x > 0. \end{cases}$$

There is a bound state of this potential when $V_0 < 0$. In this problem you are asked to solve for it.

(a) Write down the form of generic solutions of Schödinger's equation for $0 < x < a$ and $a < x$ for a state with $E < 0$.

(b) Obtain the boundary conditions (BCs) that the wavefunctions must obey at the wall ($x = 0$).

(c) Obtain the boundary conditions (BCs) that the wavefunctions must obey at $x = a$.

(d) Solve for the amplitudes in part (a) using the BCs in parts (b,c), and obtain a transcendental equation from which you could calculate the value of the state with $E < 0$ (**you do not need to solve the equation, just write it**).

3. Consider a harmonic oscillator with Hamiltonian $\hat{H} = \frac{\hat{p}^2}{2m} + \frac{1}{2}m\omega^2\hat{x}^2 = \hbar\omega\,(a^\dagger a + 1/2)$, with $a, a^\dagger$ defined as

$$a = \frac{1}{\sqrt{2}}\left(\sqrt{\frac{m\omega}{\hbar}}\,\hat{x} + i\sqrt{\frac{1}{m\hbar\omega}}\,\hat{p}\right) \quad \text{and} \quad a^\dagger = \frac{1}{\sqrt{2}}\left(\sqrt{\frac{m\omega}{\hbar}}\,\hat{x} - i\sqrt{\frac{1}{m\hbar\omega}}\,\hat{p}\right) .$$

The ground state wavefunction $\psi_0(x)$ can be obtained from the condition $a\psi_0(x) = 0$, and the excited states from

$$\psi_n(x) = \frac{1}{\sqrt{n!}}\,(a^\dagger)^n\,\psi_0(x) .$$

(a) Compute $\psi_0(x)$ and $\psi_1(x)$ explicitly.

Now we will use these results to obtain the energy levels of a particle in the following "half" harmonic potential well:

$$V(x) = \begin{cases} \infty & \text{if } x < 0, \\ \frac{1}{2}m\omega^2 x^2 & \text{if } x > 0. \end{cases}$$

(b) Write down Schödinger's equation for $x > 0$; do $\psi_0(x)$ and $\psi_1(x)$ satisfy this equation? (You do not to show one way or another explicitly, just provide a simple argument.)

(c) Write down the boundary conditions (BCs) that the wavefunctions must obey at the origin.

(d) Do $\psi_0(x)$ and $\psi_1(x)$ satisfy the BCs?

(e) What is the ground state wavefunction of the "half" harmonic oscillator.

(f) Without any calculation, by generalizing your results to other even or odd $\psi_n(x)$, determine all the energy eigenvalues of the "half" harmonic oscillator potential.