Quantum Mechanics I

Exam I

Date: Thursday, February 21, 2008, 9:30am - 11:00am

This is a CLOSED book exam. There are 3 questions. Write your name in all pages.
Use the space in the back of each page if you need additional space to write your solutions.

1. Consider two linear operators $\hat{A}$ and $\hat{B}$.
   (a) Show that the operator $\hat{O}_{\mathrm{S}} = \hat{A} + \hat{B}$ is linear.
   (b) Show that the operator $\hat{O}_{\mathrm{P}} = \hat{A}\,\hat{B}$ is also linear.
   (c) Is the operator $\hat{O}_{n,m} = \hat{x}^n + (i\hat{p})^m$, for some positive integers n, m linear? Justify your answer.
   (d) Show that the operator $\hat{O}_{1,1}$ (as define above) has a nontrivial eigenfunction $\psi(x)$ with zero eigenvalue. Compute this eigenfunction, and show that it is normalizable.

2. Consider a particle inside an infinite potential well defined by the potential $V(x) = 0$ for $0 \leq x \leq a$, and $V(x) = \infty$ otherwise. The eigenfunctions and eigenenergies are given by:

$$\psi_n(x) = \sqrt{\frac{2}{a}} \sin\left(\frac{n\pi x}{a}\right), \quad E_n = \frac{\hbar^2}{2m}\frac{\pi^2}{a^2}n^2, \quad \text{for} \quad n = 1, 2, 3, \ldots$$

Suppose one prepares an initial state

$$\Psi(x, 0) = A\left(\frac{1}{\sqrt{2}}\,\psi_1(x) + \frac{1}{\sqrt{3}}\,\psi_2(x) + \frac{1}{\sqrt{6}}\,\psi_3(x)\right)$$

(a) Determine the value of A (take A real) from the normalization of the wavefunction. (Hint: if you think a bit, you do not need to actually do *any* integral.)

(b) Write down the expression for $\Psi(x, t)$ for $t > 0$.

(c) What is the probability that a measurement of the energy yields the nth eigenenergy E_n?

(d) What is the expectation value $\langle E \rangle$ of the energy in this state at some time $t > 0$?

(e) Suppose that at some time t_{M} you measure the energy of the particle. After that measurement, what is larger: the probability P_L of finding the particle in the left hand half side ($0 < x < a/2$) of the well, or the probability P_R that it is in the right hand half side ($a/2 < x < 0$)? Justify.

3. A free particle with mass m is prepared in an initial state

$$\Psi(x,0) = A\, e^{-\frac{1}{2}ax^2}\, e^{ibx}$$

(a) Find the normalization constant A as a function of a and b.

(b) Decompose the wavefunction in the basis of momentum eigenfunctions $\psi_k(x) = \frac{e^{ikx}}{\sqrt{2\pi}}$.

(c) Write down the expression for the wavefunction at a time t; you are *not* required to evaluate the integral.

(d) Find the probability distribution for observing the particle with a momentum $p = \hbar k$.

(e) Find $\langle p \rangle$ as a function of time.