3.a Nonlinear, ordinary differential equations

We briefly address the problem of nonlinear differential equations. Since we will be dealing with evolution equations, we will use here the notation \( t \) for the independent variable, and \( x(t), y(t), \) or \( x_i(t) \) for the dependent variables.

It is useful to note that higher order ODEs can be reduced to systems of linear differential equations. Consider, for example, the equation

\[
x''(t) = F(x'(t), x(t), t)
\]  

Let us introduce \( y(t) = x'(t) \) as a separate function. Then Eq. 1 can be recast as a system of first order equations:

\[
x'(t) = y(t) \quad (2)
\]

\[
y'(t) = F(y(t), x(t), t) \quad (3)
\]

If \( F \) is a linear function of \( y(t) \) and \( x(t) \), e.g.

\[
F = -P(t)y(t) - Q(t)x(t) + R(t)
\]  

then we fall back to the linear ODEs which have been considered in the notes of linear ODEs. Otherwise the equation is nonlinear.

Systems of first order nonlinear ODEs are more general than the one in Eqs. 2 and 3. They can be of the form

\[
x'(t) \equiv \frac{dx}{dt} = G(y(t), x(t), t) \quad (5)
\]

\[
y'(t) \equiv \frac{dy}{dt} = F(y(t), x(t), t) \quad (6)
\]

If Eq. 5 can be solved to give

\[
y(t) = H(x'(t), x(t), t) \quad (7)
\]

then, by differentiating, we get

\[
y'(t) = \frac{\partial H(x', x, t)}{\partial x'} x'' + \frac{\partial H(x', x, t)}{\partial x} x' + \frac{\partial H(x', x, t)}{\partial t} \quad (8)
\]

and substituting into Eq. 6 we obtain

\[
\frac{\partial H(x', x, t)}{\partial x'} x'' + \frac{\partial H(x', x, t)}{\partial x} x' + \frac{\partial H(x', x, x)}{\partial t} - F(H(x'(t), x(t), y)) = 0 \quad (9)
\]
which is an equation involving only \( x, x', x'' \), but the procedure may not be feasible or yield manageable results, in which case one should stay with the system of coupled nonlinear, first order ODEs.

The system of equations 5, 6 can be generalized to the case where we have more than two dependent variables \( x_1, x_2, \ldots x_n \), in which it will take the form

\[
\frac{dx_i}{dt} = F_i(x_j(t), t) \quad i = 1, 2, \ldots n
\] (10)

The solutions of nonlinear ODE can exhibit quite a wide variety of behaviors and, depending on the equations, the results which can be established analytically may be limited. Very often the equations need to be treated by computational techniques. A very general result which can be established analytically, however, is that, in a region where the functions \( F_i \) and their derivatives are well defined, a set of initial values

\[
x_i(t_0) = x_{i,0}
\] (11)

determines the solution \( x_i(t) \) uniquely. As a consequence different trajectories cannot have any point in common (for a given value of \( t \).) If \( x_i(t) \) and \( \tilde{x}_i(t) \) were to coincide at some \( t = t_1 \), then using \( t_1 \) as the initial value in the evolution equations, with initial data \( x_i(t_1) = \tilde{x}_i(t_1) \), we would have \( x_i(t) = \tilde{x}_i(t) \) for all \( t \).

The space spanned by the variables \( x_i(t) \) is often referred to as the phase space. The trajectories \( x_i(t) \) span curves in phase space. Let \( \gamma \) be the curve spanned by the trajectory with initial data \( x_i(t_0) = x_{i,0} \), and \( \gamma' \) be the curve spanned by the trajectory with initial data \( x_i(t_1) = x_{i,0} \) (note the different value of the initial time, with identical initial value of the dependent variables.) If the functions \( F_i(x_j(t), t) \) in Eq. 10 have an explicit time dependence, then \( \gamma \) and \( \gamma' \) can be different. To understand this point, let us imagine that the functions \( F_i \) embody a driving force which causes that variables \( x_i \) to move in one direction at \( t = t_0 \) and in another direction at \( t = t_1 \). In this case \( \gamma \) and \( \gamma' \) would be different, and might even cross at some point. (This does not violate the uniqueness of the trajectories, because the evolution times \( t - t_0 \) and \( t - t_1 \) would be different.) However if the functions \( F_i \) do not have an explicit time dependence, then the curve \( \gamma \) spanned by the trajectory which goes through the point \( x_i = x_{i,0} \) will be independent of the time at which the trajectory goes through this point. In other words, the curves spanned by the trajectories will be time independent and also, quite importantly, will be non overlapping. (If two different curves \( \gamma \) and \( \gamma' \) had a point in common, we could take this point as the initial point for the evolution. The trajectories would then be identical and \( \gamma \) and \( \gamma' \) would have to be identical as well.)

In what follows we will restrict our considerations to the case where the functions \( F_i \) have no explicit time dependence. It is then immaterial to distinguish between the curve \( \gamma \) spanned by a trajectory and the trajectory itself (in principle the trajectory would consist of \( \gamma \) plus the specification of the time at which the variables \( x_i \) go through a definite point of \( \gamma \)) and we will refer to the curves spanned in phase space as “the trajectories.”
In the specific case of a two dimensional phase space, which we will consider in the rest of these notes, a consequence of the uniqueness of the trajectories is that the phase space is divided into non-overlapping regions, each trajectory occupying one of these regions. The word “region” must be intended in a wide sense. If a trajectory is closed, as in the case of the harmonic oscillator where, taking \( x(t) \) as the coordinate and \( y(t) \) as the momentum, the trajectories are ellipses, the “regions” consist of single closed curves. In other cases, however, the trajectories may consist of open curves with the system evolving from one point of instability to a point of attraction, or a trajectory may wind in between two limiting cycles, in which case the region occupied by the trajectory would be the domain between the two limiting cycles.

As example of an interesting nonlinear ODE in two dimensional phase space, let us consider the “Lotka-Volterra equation”, which can be used to describe the behavior of a predator-prey system. The equations are

\[
\begin{align*}
\frac{dx}{dt} &= ax - bxy \\
\frac{dy}{dt} &= dxy - cy
\end{align*}
\]  

(12)  

(13)

where \( x \) represents the number of prey, \( y \) the number of predators, and \( a, b, c, d \) are positive parameters. The rationale behind the equations is that in absence of interaction between the species (i.e. with \( b = d = 0 \)) the prey would grow exponentially (\( dx/dt = ax \) is solved by \( x(t) = x(0)e^{at} \)) and the predators would die (\( dy/dt = -cy \) is solved by \( y(t) = y(0)e^{-ct} \)). However, the rate of decay of the number of predators is reversed, leading to their growth, if \( x \) is sufficiently large to make \( dx - c > 0 \). With these equations it turns out that the number of prey will always continue to grow until the number of predators begins to grow. At some point the growth of \( y \) will lead to a reversal of the rate of growth \( a - by \) of the prey, which will start to decrease in number to the point when this will stymie the growth of the predator population which will begin to decrease as well, and the whole affair will result in a cyclic behavior.

The equations have two fixed points, obtained by demanding \( dx/dt = 0 \) and \( dy/dt = 0 \). This leads to the equations

\[
\begin{align*}
ax - bxy &= 0 \\
dxy - cy &= 0
\end{align*}
\]

(14)  

(15)

which are solved by \( x_1 = y_1 = 0 \) and \( x_2 = c/d, y_2 = a/b \). We may investigate the stability of the two fixed points by linearizing the equations around them. For the first solution, the linearized equations are obviously

\[
\begin{align*}
\frac{dx}{dt} &= ax \\
\frac{dy}{dt} &= -cy
\end{align*}
\]

(16)  

(17)
which is unstable in the direction of growing $x$. So the equations tell us that, with zero predators, a minimal number of prey will lead away from the fixed point.

In order to linearize the second solution let us define new variables $\delta x = x - c/d$, $\delta y = y - a/b$. Substituting into the equations these take the form

$$\frac{d\delta x}{dt} = a\left(\delta x + \frac{c}{d}\right) - b\left(\delta x + \frac{c}{d}\right)\left(\delta y + \frac{a}{b}\right)$$

(18)

$$\frac{d\delta y}{dt} = d\left(\delta x + \frac{c}{d}\right)\left(\delta y + \frac{a}{b}\right) - c\left(\delta y + \frac{a}{b}\right)$$

(19)

or, keeping only terms of the first order in $\delta x, \delta y$,

$$\frac{d\delta x}{dt} = \frac{bc}{d} \delta y$$

(20)

$$\frac{d\delta y}{dt} = \frac{ad}{b} \delta x$$

(21)

These are the equations of a harmonic oscillators with angular frequency $\frac{ac}{\sqrt{bd}}$, which shows that the second fixed point is stable. A small displacement from the fixed point will lead to oscillations around it.

While for bigger displacements from the fixed point an analytic solution to the equations of motion cannot be given, it is possible however to find the shape the trajectory. Indeed by dividing Eq. 13 by Eq. 12 we find

$$\frac{dy}{dx} = \frac{dxy - cy}{ax - bxy}$$

(22)

or

$$(ax - bxy) dy = (dxy - cy) dx$$

(23)

By dividing by $xy$ this gives

$$\left(\frac{a}{y} - bx\right) dy = \left(dx - cy\right) dx$$

(24)

and

$$\left(\frac{a}{y} - b\right) dy = \left(dx - \frac{c}{x}\right) dx$$

(25)

which can be integrated to give

$$a \log y - by = dx - c \log x + K$$

(26)

where $K$ is a constant depending on the initial conditions.

The Lotka-Volterra equation can be generalized to more than two species and to include a variety of interactions among the the species. With quadratic interaction terms, the most general form of the equations would be

$$\frac{dx_i(t)}{dt} = a_i x_i(t) + \sum_j A_{ij} x_i(t) x_j(t)$$

(27)
where the vector $\vec{a}$ describes the behavior (growth or decline) of the species without mutual interactions, while the matrix $A$ describes the interactions among species. The solutions can exhibit a wide variety of behaviors, including fixed points, limiting cycles and chaos, determined by the parameters in $\vec{a}$ and $A$. 