Useful trigonometric formulae:

From Euler’s formula and

\[ e^{i(\alpha + \beta)} = e^{i\alpha}e^{i\beta} \]

one easily obtains

\[ \sin(\alpha + \beta) = \sin \alpha \cos \beta + \cos \alpha \sin \beta \]  \hspace{1cm} (1)

\[ \cos(\alpha + \beta) = \cos \alpha \cos \beta - \sin \alpha \sin \beta \]  \hspace{1cm} (2)

Hyperbolic functions:

\[ \cosh x = \frac{e^x + e^{-x}}{2} \]

\[ \sinh x = \frac{e^x - e^{-x}}{2} \]

\[ \tanh x = \frac{\sinh x}{\cosh x} \]

\[ \cosh x \to \infty \text{ for } x \to \pm \infty \]

\[ \sinh x \to \pm \infty \text{ for } x \to \pm \infty \]

\[ \tanh x \to \pm 1 \text{ for } x \to \pm \infty \]

Hyperbolic functions satisfy many relations similar to the relations among trigonometric functions, e.g.

\[ \cosh(x + y) = \cosh x \cosh y + \sinh x \sinh y \]

\[ \sinh(x + y) = \sinh x \cosh y + \cosh x \sinh y \]

\[ \tanh(x + y) = \frac{\tanh x + \tanh y}{1 + \tanh x \tanh y} \]

etc.

Power series expansions

\[ \cosh x = 1 + \frac{x^2}{2} + \frac{x^4}{4!} + \ldots \]

\[ \sinh x = x + \frac{x^3}{3!} + \frac{x^5}{5!} + \ldots \]
Relation to sine and cosine of imaginary arguments:

\[
\begin{align*}
\cos ix &= 1 - \frac{(ix)^2}{2} + \frac{(ix)^4}{4!} + \cdots = 1 + \frac{x^2}{2} + \frac{x^4}{4!} + \cdots = \cosh x \\
\sin ix &= ix - \frac{(ix)^3}{3!} + \frac{(ix)^5}{5!} + \cdots = ix + \frac{x^3}{3!} + \frac{x^5}{5!} + \cdots = i \sinh x
\end{align*}
\]

And thus, for example (cfr. Eqs. 1, 2)

\[
\begin{align*}
\sin(x + iy) &= \sin x \cos iy + \cos x \sin iy = \sin x \cosh y + i \cos x \sinh y \\
\cos(x + iy) &= \cos x \cos iy - \sin x \sin iy = \cos x \cosh y - i \sin x \sinh y
\end{align*}
\]
**Sum of a series by a contour integral:**

Exercise: sum the series

\[ S = \sum_{n=1}^{\infty} \frac{(-1)^{n-1}}{n^2} \]

by the method of residues.

We need to find a function that has poles at \( n = 1, 2, 3 \ldots \) with residue \((-1)^{n-1}/n^2\).

Let us extend first the series to the negative numbers:

\[ S = \frac{1}{2} \sum_{n=-\infty, n \neq 0}^{\infty} \frac{(-1)^{n-1}}{n^2} \]

Now

\[ s(z) = \frac{1}{\sin \pi z} \]

has simple poles at all integers \( n \) with residue \((-1)^n/\pi\).

Thus we will succeed if we find a contour \( \gamma \) which circles around all the poles of \( s(z) \) in clockwise direction and calculate

\[ I = \frac{1}{4i} \oint_{\gamma} \frac{1}{z^2 \sin \pi z} \, dz \]

Consider the contour integral \( \gamma \) shown in Fig. 1, where we will eventually let the two big semicircles go to infinity. When \( z \rightarrow \pm i \infty \), \( \sin z \rightarrow e^{\pm|z|}/2 \) and, together with the denominator \( z^2 \), this guarantees that the contribution to the integral from the semicircles at infinity vanishes. So the only contribution to \( \oint_{\gamma} \) comes from the integrals on small circles around the poles of

\[ f(z) = \frac{1}{z^2 \sin \pi z} \]

which are equal to

\[-\text{Res}(f(z), n) = \frac{(-1)^n}{\pi n^2} = \frac{(-1)^{n-1}}{\pi n^2} \]

(The negative sign in front of Res is because the path along the circles are in clockwise direction.) Thus

\[ I = \frac{1}{4i} \sum_{n=-\infty, n \neq 0}^{\infty} 2\pi i \text{Res}(f(z), n) = \frac{1}{2} \sum_{n=-\infty, n \neq 0}^{\infty} \frac{(-1)^{n-1}}{n^2} = S \]
On the other hand, the function $f(z)$ is regular inside the contour $\gamma$ apart from the singularity at $z = 0$. Thus, from the theorem of residues, we get

$$I = \frac{1}{4\pi} 2\pi i \text{Res}(f(z),0) = \frac{\pi}{2} \text{Res}(f(z),0)$$

To find the residue of $f(z)$ at 0 we expand

$$f(z) = \frac{1}{z^2 \sin \pi z} = \frac{1}{z^2 [\pi z - (\pi z)^3/6 + \ldots]} = \frac{1}{\pi z^3} \frac{1}{1 - (\pi z)^2/6 + \ldots} = \frac{1}{\pi z^3} \frac{1 + (\pi z)^2/6 + \ldots}{\pi z^3} = \frac{1}{\pi z^3} + \frac{\pi}{6z} + \ldots$$

which gives

$$\text{Res}(f(z),0) = \frac{\pi}{6}$$
So, finally, we obtain

\[ I = \frac{\pi}{2} \text{Res}(f(z), 0) = \frac{\pi^2}{12} \]

This result can be checked numerically:

Enter the maximum n or any character to quit: 100
n= 97, sum= 0.822520, exact= 0.822467, diff= 0.000053
n= 98, sum= 0.822416, exact= 0.822467, diff= -0.000052
n= 99, sum= 0.822518, exact= 0.822467, diff= 0.000050
n= 100, sum= 0.822418, exact= 0.822467, diff= -0.000050

Enter the maximum n or any character to quit: 1000
n= 997, sum= 0.822468, exact= 0.822467, diff= 0.000000
n= 998, sum= 0.822467, exact= 0.822467, diff= -0.000001
n= 999, sum= 0.822468, exact= 0.822467, diff= 0.000000
n= 1000, sum= 0.822467, exact= 0.822467, diff= -0.000001

And how about the series \( S_2 = \sum_{n=1}^{\infty} \frac{1}{n^2} \)?

We must find a way to eliminate the alternation of signs in the residues of \( 1/\sin \pi z \).

Let us try

\[ I_2 = -\frac{1}{4\pi} \oint_{\gamma} \frac{\cos \pi z}{z^2 \sin \pi z} \, dz \]

With \( \cos \pi n = (-1)^n \) and, now,

\[ f(z) = \frac{\cos \pi z}{z^2 \sin \pi z} \]

summing over the residues at integer, non-zero, \( z \) will give us

\[ I_2 = -\frac{1}{4\pi} \sum_{n=-\infty, n \neq 0}^{\infty} 2\pi i \left[ -\text{Res}(f(z), n) \right] = \frac{1}{2} \sum_{n=-\infty, n \neq 0}^{\infty} \frac{1}{n^2} = S_2 \]

But can we still neglect the semicircles at infinity?

The answer is yes, because for \( z \to \pm i\infty \) both \( \cos z \) at numerator and \( \sin z \) at denominator behave as \( \sim e^{\pm |z|}/2 \) and so their ratio tends to 1, but the factor \( z^2 \) at denominator saves the day and guarantees that the contribution to the integral from the two semicircles goes to zero. Closing again the contour around the pole at \( z = \), or, equivalently, from the theorem of residues, we get again

\[ I_2 = -\frac{1}{4\pi} 2\pi i \text{Res}(f(z), 0) = -\frac{\pi}{2} \text{Res}(f(z), 0) \]
but now with

\[ f(z) = \frac{\cos \pi z}{z^2 \sin \pi z} = \frac{(1 - (\pi z)^2/2 + \ldots)(1 + (\pi z)^2/6 + \ldots)}{z^3} = \frac{1 - (\pi z)^2/3 + \ldots}{z^3} = \frac{1}{\pi z^3} - \frac{\pi}{3z} + \ldots \]

Thus

\[ \text{Res}(f(z), 0) = -\frac{\pi}{3} \]

and

\[ I_2 = -\frac{\pi}{2} \text{Res}(f(z), 0) = \frac{\pi^2}{6} \]

This result can also be verified numerically (this series converges more slowly because all the terms have the same sign:)

Enter the maximum n or any character to quit: 1000
n= 997, sum= 1.643932, exact= 1.644934, diff= -0.001003
n= 998, sum= 1.643933, exact= 1.644934, diff= -0.001002
n= 999, sum= 1.643934, exact= 1.644934, diff= -0.001001
n= 1000, sum= 1.643935, exact= 1.644934, diff= -0.001000

Enter the maximum n or any character to quit: 10000
n= 9997, sum= 1.644834, exact= 1.644934, diff= -0.000100
n= 9998, sum= 1.644834, exact= 1.644934, diff= -0.000100
n= 9999, sum= 1.644834, exact= 1.644934, diff= -0.000100
n= 10000, sum= 1.644834, exact= 1.644934, diff= -0.000100

Enter the maximum n or any character to quit: 100000
n= 99997, sum= 1.644933, exact= 1.644934, diff= -0.000001
n= 99998, sum= 1.644933, exact= 1.644934, diff= -0.000001
n= 99999, sum= 1.644933, exact= 1.644934, diff= -0.000001
n= 100000, sum= 1.644933, exact= 1.644934, diff= -0.000001