1. Vector calculus

Cartesian coordinates.

\[ \vec{r} = x \hat{i} + y \hat{j} + z \hat{k} \]  

The unit vectors \( \hat{i}, \hat{j}, \hat{k} \) are orthonormal.

Change of frame. Rotation around the \( z \)-axis by \( \phi \): new frame \( \hat{i}', \hat{j}', \hat{k}' \) are:

\[ \hat{i}' = \hat{j}, \quad \hat{j}' = -\hat{i}, \quad \hat{k}' = \hat{k} \]

New coordinates \( x', y', z' \)

\[
\begin{pmatrix}
  x' \\
  y' \\
  z'
\end{pmatrix}
= R
\begin{pmatrix}
  x \\
  y \\
  z
\end{pmatrix}
\]

with

\[
R = \begin{pmatrix}
\cos \phi & \sin \phi & 0 \\
-\sin \phi & \cos \phi & 0 \\
0 & 0 & 1
\end{pmatrix}
\]

Sign check: with \( \phi = \pi/2 \), \( x = 0, y = 1, z = 0 \) becomes \( x' = 1, y' = 0, z' = 0 \) o.k. (*)

\( \vec{r} \) is unchanged:

\[ \vec{r} = x \hat{i} + y \hat{j} + z \hat{k} = x' \hat{i}' + y' \hat{j}' + z' \hat{k}' \]  

If we keep the original basis, then

\[ \vec{r} \rightarrow \vec{r}' = x' \hat{i} + y' \hat{j} + z' \hat{k} \]  

and \( \vec{r} \) rotates by \( -\phi \) (check from (*): now \( \vec{r} = \hat{j} \rightarrow \vec{r}' = \hat{i} \).)

A general 3-d rotation can be obtained by a sequence of 2-d rotations around different axes (cfr. Euler angles.)

For a general rotation \( R \), with

\[ \vec{w} = R \vec{v} \]  

we have

\[ \vec{w} \cdot \vec{w} = (R \vec{v}) \cdot (R \vec{v}) = \vec{v} \cdot R^\text{tr} R \vec{v} = \vec{v} \cdot \vec{v} \]  

hence

\[ R^\text{tr} R = I, \quad \text{or} \quad R^\text{tr} = R^{-1} \]  

\( R \) is called orthogonal. With complex vectors conservation of the length requires

\[ R^\dagger R \equiv (R^\text{tr})^* R = I, \quad \text{or} \quad R^\dagger = R^{-1} \]  

and \( R \) is called unitary.
Scalar and vector fields.

Scalar field: \( F(x, y, z) \).

Vector field \( \vec{v}(x, y, z) = (v_x(x, y, z), v_y(x, y, z), v_z(x, y, z)) \).

Gradient:
\[
\text{grad } F \equiv \nabla F = \left( \frac{\partial F}{\partial x}, \frac{\partial F}{\partial y}, \frac{\partial F}{\partial z} \right)
\] (10)

Nabla operator:
\[
\nabla F = \mathbf{\hat{i}} \frac{\partial}{\partial x} + \mathbf{\hat{j}} \frac{\partial}{\partial y} + \mathbf{\hat{k}} \frac{\partial}{\partial z}
\] (11)

With \( \vec{r} \rightarrow \vec{r} + d\vec{r} \):
\[
dF = \nabla F \cdot d\vec{r}
\] (12)

Directional derivative:
\[
\vec{r} \equiv \vec{r}(s) \text{ (s is the arc length along the curve)}
\]
\[
ds = |d\vec{r}|
\] (13)
\[
\frac{dF}{ds} = \nabla F \cdot \frac{d\vec{r}}{ds}
\] (14)

(note: \( |d\vec{r}/ds| = 1 \))

With \( \gamma \equiv \vec{r}(s) \), \( \vec{r}(s_a) = \vec{r}_a \), \( \vec{r}(s_b) = \vec{r}_b \),
\[
\int_{\gamma} \nabla F \cdot d\vec{r} = \int_{s_a}^{s_b} \frac{dF}{ds} ds = F(\vec{r}_b) - F(\vec{r}_a)
\] (15)

independently of \( \gamma \) (within a domain where \( F \) is regular and single valued.)

Divergence:
\[
\text{div } \vec{v} \equiv \nabla \cdot \vec{v} = \frac{\partial v_x}{\partial x} + \frac{\partial v_y}{\partial y} + \frac{\partial v_z}{\partial z}
\] (16)

Laplace operator, or Laplacian
\[
\Delta \equiv \nabla^2 \equiv \nabla \cdot \nabla \ (\equiv \text{div grad})
\] (17)
\[
\Delta F = \nabla \cdot \nabla F
\] (18)

Gauss’ theorem:
\[
\int_V \nabla \cdot \vec{v} \ dV \equiv \int_V \nabla \cdot \vec{v} \ dx \ dy \ dz = \int_{\Sigma} \vec{v} \cdot \mathbf{n} \ d\sigma
\] (19)

where \( V \) is a closed volume bounded by the surface \( \Sigma \), \( \mathbf{n} \) is the outward normal to the surface, \( d\sigma \) is the area element of the surface, and \( \vec{v} \) is assumed to be regular inside \( V \).
Curl:
\[
\text{curl } \vec{v} \equiv \nabla \times \vec{v} = \left( \frac{\partial v_z}{\partial y} - \frac{\partial v_y}{\partial z}, \frac{\partial v_x}{\partial z} - \frac{\partial v_z}{\partial x}, \frac{\partial v_y}{\partial x} - \frac{\partial v_x}{\partial y} \right)
\] (20)

Note:
\[
\nabla \cdot \nabla \times \vec{v} = 0
\] (21)

(div curl \( \vec{v} = 0 \))

Stokes’ theorem:
\( \Sigma \) is a surface bound by the oriented curve \( \gamma = \vec{r}(s) \), \( \hat{n} \) is the outward normal to the surface, \( d\sigma \) is the area element of the surface, \( s \) is the length along \( \gamma \), the orientation of \( \gamma \) is counterclockwise as seen from \( \hat{n} \), \( \vec{v} \) is assumed to be regular in a domain enclosing \( \Sigma \), then:
\[\int_\gamma \vec{v} \cdot d\vec{r} ds = \int_\Sigma (\nabla \times \vec{v}) \cdot \hat{n} d\sigma \] (22)

Note: as a consequence of Eq. 21 and Gauss’ theorem, the integral in the r.h.s. of Eq. 22 is independent of the specific surface bounded by \( \gamma \), as it should.

If \( \text{curl } \vec{v} = 0 \) over a simply connected domain\(^1\), then the integral
\[\Phi = \int_\gamma \vec{v} \cdot d\vec{r} \] (23)

over any open curve \( \gamma \) from \( \vec{r}_a \) to \( \vec{r}_b \) within this domain depends only on the end points of \( \gamma \) but not on the path leading from \( \vec{r}_a \) to \( \vec{r}_b \)
\[\Phi = \Phi(\vec{r}_a, \vec{r}_b) \] (24)

In particular, if one takes a fixed \( \vec{r}_a = \vec{r}_0 \) and varies only \( \vec{r} = \vec{r}_b \), one obtains a “potential function” \( \Phi(\vec{r}) \) with the property that
\[\vec{v} = \nabla \Phi(\vec{r}) \] (25)

The potential is only defined up to an additive constant (corresponding to different choices of the fixed point \( \vec{r}_a \).\(^2\))

Vector operators in 2-d
(This will be important in the study of functions of complex variables.)
Consider a situation where there is no \( z \)-dependence and \( v_z = 0 \). Thus we effectively have a 2-d system:
\[F = F(x, y) \] (26)

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\(^1\)i.e. a domain where any closed curve can be smoothly contracted to a single point

\(^2\)Often the potential function is defined with a negative sign, for example the electric field is given by \( \vec{E} = -\nabla \Phi \), \( \Phi \) being the electrostatic potential.
\( \vec{v} = (v_x(x, y), v_y(x, y)) \)  

Then:

\[ \nabla F = \left( \frac{\partial F}{\partial x}, \frac{\partial F}{\partial y} \right) \]  

\[ \text{div } \vec{v} \equiv \nabla \cdot \vec{v} = \frac{\partial v_x}{\partial x} + \frac{\partial v_y}{\partial y} \]  

Like \( \text{div } \vec{v} \), \( \text{curl } \vec{v} \) is also a scalar

\[ \text{curl } \vec{v} \equiv \nabla \times \vec{v} = \frac{\partial v_y}{\partial x} - \frac{\partial v_x}{\partial y} \]  

If \( \text{curl } \vec{v} = 0 \) over a simply connected domain \( D \), then within \( D \) one can find a function \( \Phi(x, y) \) with

\[ v_x = \frac{\partial \Phi}{\partial x} \]  
\[ v_y = \frac{\partial \Phi}{\partial y} \]  

If \( \text{div } \vec{v} = 0 \), define

\[ u_y = v_x \]  
\[ u_x = -v_y \]  

Then

\[ \text{curl } \vec{u} = \frac{\partial u_y}{\partial x} - \frac{\partial u_x}{\partial y} = \frac{\partial v_x}{\partial x} + \frac{\partial v_y}{\partial y} = 0 \]  

and one can find a function \( \Psi(x, y) \) with

\[ v_x = u_y = \frac{\partial \Psi}{\partial y} \]  
\[ v_y = -u_x = -\frac{\partial \Psi}{\partial x} \]  

If both \( \text{curl } \vec{v} = 0 \) and \( \text{div } \vec{v} = 0 \), then \( \Phi \) and \( \Psi \) satisfy the Laplace equation

\[ \Delta \Phi = 0 \]  
\[ \Delta \Psi = 0 \]  

Curvilinear coordinates
exemplified by spherical coordinates \( r, \theta, \phi \).

\[ x = r \sin \theta \cos \phi \]  
\[ y = r \sin \theta \sin \phi \]  
\[ z = r \cos \theta \]
\[ \vec{r} = \vec{r}(r, \theta, \phi) \]
\[ d\vec{r} = \frac{\partial \vec{r}}{\partial r} dr + \frac{\partial \vec{r}}{\partial \theta} d\theta + \frac{\partial \vec{r}}{\partial \phi} d\phi = h_r \hat{r} dr + h_\theta \hat{\theta} d\theta + h_\phi \hat{\phi} d\phi \]
\[ dF = \text{grad} F \cdot d\vec{r} = \text{grad} F \cdot (h_r \hat{r} dr + h_\theta \hat{\theta} d\theta + h_\phi \hat{\phi} d\phi) \]
\[ \text{but also} \]
\[ dF = \frac{\partial F}{\partial r} dr + \frac{\partial F}{\partial \theta} d\theta + \frac{\partial F}{\partial \phi} d\phi \]
\[ \text{hence} \]
\[ \text{grad} F = \frac{1}{h_r} \frac{\partial F}{\partial r} \hat{r} + \frac{1}{h_\theta} \frac{\partial F}{\partial \theta} \hat{\theta} + \frac{1}{h_\phi} \frac{\partial F}{\partial \phi} \hat{\phi} \]
\[ \text{With spherical coordinates } h_r = 1, h_\theta = r, h_\phi = r \sin \theta \text{ and} \]
\[ \text{grad} F = \frac{\partial F}{\partial r} \hat{r} + \frac{1}{r} \frac{\partial F}{\partial \theta} \hat{\theta} + \frac{1}{r \sin \theta} \frac{\partial F}{\partial \phi} \hat{\phi} \]
\[ \vec{v} = v_r(r, \theta, \phi) \hat{r} + v_\theta(r, \theta, \phi) \hat{\theta} + v_\phi(r, \theta, \phi) \hat{\phi}. \]
\[ \text{Calculate } \text{div} \vec{v}: \]
\[ \text{Use Gauss’ theorem. Consider the flux of } \vec{v} \text{ through the faces of a cube with sides oriented along } \hat{r}, \hat{\theta}, \hat{\phi} \text{ and lengths } h_r dr, h_\theta d\theta, h_\phi d\phi. \]
\[ \Phi_{\hat{\phi}} = h_\theta h_\phi v_r d\theta d\phi \]
\[ \text{As } r \text{ increases by } dr \text{ along the } \hat{r} \text{ side, the differential of this flux, i.e. the net outgoing flux will be} \]
\[ d\Phi_{\hat{\phi}} = \frac{\partial(h_\theta h_\phi v_r)}{\partial r} dr d\theta d\phi \]
\[ \text{Adding similar contributions through the } \hat{\phi}, \hat{r} \text{ and } \hat{\theta}, \hat{\phi} \text{ faces we find a total outgoing flux} \]
\[ d\Phi = \left( \frac{\partial(h_\theta h_\phi v_r)}{\partial r} + \frac{\partial(h_\phi h_r v_\theta)}{\partial \theta} + \frac{\partial(h_r h_\theta v_\phi)}{\partial \phi} \right) dr d\theta d\phi \]
On the other hand by Gauss’ theorem $d\Phi$ must be equal to the integral of $\text{div} \, \vec{v}$ over the cube, i.e.

$$d\Phi = h_r h_\theta h_\phi (\text{div} \, \vec{v}) \, dr \, d\theta \, d\phi$$

Comparing Eqs. 53, 54 we get

$$\text{div} \, \vec{v} = \frac{1}{h_r h_\theta h_\phi} \left( \frac{\partial (h_\theta h_\phi \, v_r)}{\partial r} + \frac{\partial (h_\phi h_r \, v_\theta)}{\partial \theta} + \frac{\partial (h_r h_\theta \, v_\phi)}{\partial \phi} \right)$$

Specifically

$$\text{div} \, \vec{v} = \frac{1}{r^2 \sin \theta} \left( \frac{\partial (r^2 \sin \theta \, v_r)}{\partial r} + \frac{\partial (r \sin \theta \, v_\theta)}{\partial \theta} + \frac{\partial (r \, v_\phi)}{\partial \phi} \right)$$

$$= \frac{\partial v_r}{\partial r} + \frac{2}{r} \frac{v_r}{r} + \frac{1}{r} \frac{\partial v_\theta}{\partial \theta} + \cot \theta \frac{v_\theta}{r} + \frac{1}{r \sin \theta} \frac{\partial v_\phi}{\partial \phi}$$

Proceeding in a similar fashion from Stokes’ theorem one finds

$$\text{curl} \, \vec{v} = \frac{1}{r^2 \sin \theta} \left( \frac{\partial (r \sin \theta \, u_\phi)}{\partial \theta} - \frac{\partial (r \sin \theta \, u_\theta)}{\partial \phi} \right) \hat{r} +$$

$$\frac{1}{r \sin \theta} \left( \frac{\partial u_r}{\partial \phi} - \frac{\partial (r \sin \theta \, u_\theta)}{\partial r} \right) \hat{\theta} + \frac{1}{r} \left( \frac{\partial (r \sin \theta \, u_\phi)}{\partial \theta} - \frac{\partial u_r}{\partial \theta} \right) \hat{\phi}$$

The Laplacian:

$$\text{div grad} \, F =$$

$$\left( \frac{\partial}{\partial r} + \frac{2}{r} \right) \left( \frac{\partial F}{\partial r} \right) + \left( \frac{1}{r} \frac{\partial}{\partial \theta} + \cot \theta \right) \left( \frac{1}{r} \frac{\partial F}{\partial \theta} \right) + \left( \frac{1}{r \sin \theta} \frac{\partial}{\partial \phi} \right) \left( \frac{1}{r \sin \theta} \frac{\partial F}{\partial \phi} \right) =$$

$$\frac{1}{r^2} \frac{\partial}{\partial r} \left( r^2 \frac{\partial F}{\partial r} \right) + \frac{1}{r^2 \sin \theta} \frac{\partial}{\partial \theta} \left( \sin \theta \frac{\partial F}{\partial \theta} \right) + \frac{1}{r^2 \sin^2 \theta} \frac{\partial^2 F}{\partial \phi^2}$$