Hyperbolic functions:

\[
\begin{align*}
\cosh x &= \frac{e^x + e^{-x}}{2} \\
\sinh x &= \frac{e^x - e^{-x}}{2} \\
\tanh x &= \frac{\sinh x}{\cosh x}
\end{align*}
\]

\(\cosh x \to \infty\) for \(x \to \pm \infty\)
\(\sinh x \to \pm \infty\) for \(x \to \pm \infty\)
\(\tanh x \to \pm 1\) for \(x \to \pm \infty\)

Hyperbolic functions satisfy many relations similar to the relations among trigonometric functions, e.g.

\[
\begin{align*}
\cosh(x + y) &= \cosh x \cosh y + \sinh x \sinh y \\
\sinh(x + y) &= \sinh x \cosh y + \cosh x \sinh y \\
\tanh(x + y) &= \frac{\tanh x + \tanh y}{1 + \tanh x \tanh y}
\end{align*}
\]

etc.

Power series expansions

\[
\begin{align*}
\cosh x &= 1 + \frac{x^2}{2!} + \frac{x^4}{4!} + \ldots \\
\sinh x &= x + \frac{x^3}{3!} + \frac{x^5}{5!} + \ldots
\end{align*}
\]

Relation to sine and cosine of imaginary arguments:

\[
\begin{align*}
\cos ix &= 1 - \frac{(ix)^2}{2!} + \frac{(ix)^4}{4!} + \cdots = 1 + \frac{x^2}{2!} + \frac{x^4}{4!} + \cdots = \cosh x \\
\sin ix &= ix - \frac{(ix)^3}{3!} + \frac{(ix)^5}{5!} + \cdots = ix + \frac{x^3}{3!} + \frac{x^5}{5!} + \cdots = i \sinh x
\end{align*}
\]

And thus, for example

\[
\sin(x + iy) = \sin x \cos iy + \cos x \sin iy = \sin x \cosh y + i \cos x \sinh y
\]

etc.
Exercise: use a contour integral to calculate the series

\[ S = \sum_{n=1}^{\infty} \frac{(-1)^{n-1}}{n^2} \]

by the method of residues.

We need to find a function that has poles at \( n = 1, 2, 3 \ldots \) with residue \((-1)^{n-1}/n^2\).

Let us extend first the series to the negative numbers:

\[ S = \frac{1}{2} \sum_{n=-\infty, n \neq 0}^{\infty} \frac{(-1)^{n-1}}{n^2} \]

Now

\[ s(z) = \frac{1}{\sin \pi z} \]

has simple poles at all integers \( n \) with residue \((-1)^n\).

Thus we will succeed if we find a contour \( \gamma \) which encloses all the poles of \( s(z) \) and calculate

\[ I = \frac{1}{4i} \oint_{\gamma} \frac{1}{z^2 \sin \pi z} \, dz \]

Consider the contour integral \( \gamma \) shown in Fig. 1, where we will eventually let the two big semicircles go to infinity.

When \( z \to \pm i \infty \), \( \sin z \to e^{|z|}/2 \) and, together with the denominator \( z^2 \), this guarantees that the contribution to the integral from the semicircles at infinity vanishes. So the only contribution to \( \oint_{\gamma} \) comes from the small circles around the poles of

\[ f(z) = \frac{1}{z^2 \sin \pi z} \]

which are equal to

\[ -\text{Res}(f(z), n) = -\frac{(-1)^n}{\pi n^2} = \frac{(-1)^{n-1}}{\pi n^2} \]

(The negative sign in front of Res is because the path along the circles are in clockwise direction.) Thus

\[ I = \frac{1}{4i} \sum_{n=-\infty, n \neq 0}^{\infty} 2\pi i \text{Res}(f(z), n) = \frac{1}{2} \sum_{n=-\infty, n \neq 0}^{\infty} \frac{(-1)^{n-1}}{n^2} = S \]

On the other hand, the function \( f(z) \) is regular inside the contour \( \gamma \) apart form the singularity at \( z = 0 \). Thus, from the theorem of residues, we get

\[ I = \frac{1}{4i} 2\pi i \text{Res}(f(z), 0) = \frac{\pi}{2} \text{Res}(f(z), 0) \]
To find the residue of $f(z)$ at 0 we expand

$$f(z) = \frac{1}{z^2 \sin \pi z} = \frac{1}{z^2 [\pi z - (\pi z)^3/6 + \ldots]} = \frac{1}{\pi z^3} \left( 1 - \frac{(\pi z)^2}{6} + \ldots \right) = \frac{1}{\pi z^3} \frac{1 + (\pi z)^2/6 + \ldots}{\pi z^3} = \frac{1}{\pi z^3} + \frac{\pi}{6z} + \ldots$$

which gives

$$\text{Res}(f(z), 0) = \frac{\pi}{6}$$

so, finally, we obtain

$$I = \frac{\pi}{2} \text{Res}(f(z), 0) = \frac{\pi^2}{12}$$

This result can be checked numerically:
Enter the maximum n or any character to quit: 100
n = 97, sum= 0.822520, exact= 0.822467, diff= 0.000053
n = 98, sum= 0.822416, exact= 0.822467, diff= -0.000052
n = 99, sum= 0.822518, exact= 0.822467, diff= 0.000050
n = 100, sum= 0.822418, exact= 0.822467, diff= -0.000050
Enter the maximum n or any character to quit: 1000
n = 997, sum= 0.822468, exact= 0.822467, diff= 0.000000
n = 998, sum= 0.822467, exact= 0.822467, diff= -0.000001
n = 999, sum= 0.822468, exact= 0.822467, diff= 0.000000
n = 1000, sum= 0.822467, exact= 0.822467, diff= -0.000001

And how about the series $S_2 = \sum_{n=1}^{\infty} 1/n^2$?
We must find a way to eliminate the alternation of signs in the residues of $1/\sin \pi z$.
Let us try

$$I_2 = -\frac{1}{4l} \oint_{\gamma} \frac{\cos \pi z}{z^2 \sin \pi z} \, dz$$

With $\cos \pi n = (-1)^n$ and, now,

$$f(z) = \frac{\cos \pi z}{z^2 \sin \pi z}$$

summing over the residues at integer, non-zero, $z$ will give us

$$I_2 = -\frac{1}{4l} \sum_{n=\infty}^{\infty} 2\pi i \left[ -\text{Res}(f(z), n) \right] = \frac{1}{2} \sum_{n=\infty}^{\infty} \frac{1}{n^2} = S_2$$

But can we still neglect the semicircles at infinity?
The answer is yes, because for $z \to \pm i\infty$ both $\cos z$ at numerator and $\sin z$ at denominator
behave as $\sim e^{\pm z}/2$ and so their ratio tends to 1, but the factor $z^2$ at denominator saves the
day and guarantees that the contribution to the integral from the two semicircles goes to zero. Closing again the contour around the pole at $z =$, or, equivalently, from the theorem of residues, we get again

$$I_2 = -\frac{1}{4l} 2\pi i \text{Res}(f(z), 0) = -\frac{\pi}{2} \text{Res}(f(z), 0)$$

but now with

$$f(z) = \frac{\cos \pi z}{z^2 \sin \pi z} = \frac{(1 - (\pi z)^2/2 + \ldots)(1 + (\pi z)^2/6 + \ldots)}{z^3} \quad z^3 = \frac{1 - (\pi z)^2/3 + \ldots}{z^3} = \frac{1}{\pi z^3} - \frac{\pi}{3z} + \ldots$$
Thus

$$\text{Res}(f(z), 0) = -\frac{\pi}{3}$$

and

$$I_2 = -\frac{\pi}{2} \text{Res}(f(z), 0) = \frac{\pi^2}{6}$$

This result can also be verified numerically (this series converges more slowly because all the terms have the same sign):

Enter the maximum n or any character to quit: 1000
n= 997, sum= 1.643932, exact= 1.644934, diff= -0.001003
n= 998, sum= 1.643933, exact= 1.644934, diff= -0.001002
n= 999, sum= 1.643934, exact= 1.644934, diff= -0.001001
n= 1000, sum= 1.643935, exact= 1.644934, diff= -0.001000
Enter the maximum n or any character to quit: 10000
n= 9997, sum= 1.644834, exact= 1.644934, diff= -0.000100
n= 9998, sum= 1.644834, exact= 1.644934, diff= -0.000100
n= 9999, sum= 1.644834, exact= 1.644934, diff= -0.000100
n= 10000, sum= 1.644834, exact= 1.644934, diff= -0.000100
Enter the maximum n or any character to quit: 1000000
n= 999997, sum= 1.644933, exact= 1.644934, diff= -0.000001
n= 999998, sum= 1.644933, exact= 1.644934, diff= -0.000001
n= 999999, sum= 1.644933, exact= 1.644934, diff= -0.000001
n= 1000000, sum= 1.644933, exact= 1.644934, diff= -0.000001