PY 501 Mathematical Physics


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Definitions

PDEs are differential equations with more than one independent variable and one or more dependent variables. We will exemplify many of our considerations with systems with two independent variables, $x$ and $y$, and one dependent variable $\phi(x, y)$, or $x$ and $t$, and $\phi(x, t)$. But we will also consider equations with three or four independent variable, e.g. $x, y, z$ or $x, y, z, t$.

In most cases PDEs involve second order derivatives and we will restrict our considerations to these. Some interesting phenomena are described by first order partial derivatives, and we should also remember that second order equations can be reduced to systems of first order equations by defining first order derivatives as new dependent variables. Equations with partial derivatives of order higher than two are seldom encountered.

It will be convenient to denote partial derivatives with a suffix, as follows:

\[
\phi_x(x, y) \equiv \frac{\partial \phi(x, y)}{\partial x}, \quad \phi_y(x, y) \equiv \frac{\partial \phi(x, y)}{\partial y} \tag{1}
\]

\[
\phi_{xx}(x, y) \equiv \frac{\partial^2 \phi(x, y)}{\partial x^2}, \quad \phi_{xy}(x, y) \equiv \frac{\partial^2 \phi(x, y)}{\partial x \partial y} \tag{3}
\]

\[
\phi_{yy}(x, y) \equiv \frac{\partial^2 \phi(x, y)}{\partial y^2} \tag{5}
\]

and so on.

The second order PDEs one encounters in physics and engineering are, quite generally, linear in the second order derivatives with coefficients which only depend on the independent coordinates. So, exemplifying again with two independent variables, they are of the form

\[
a(x, y)\phi_{xx} + 2b(x, y)\phi_{xy} + c(x, y)\phi_{yy} + r(\phi_x, \phi_y, \phi, x, y) = 0 \tag{6}
\]

If the last term, $r(\phi, \phi_x, \phi_y, x, y)$ in Eq. 6 is also linear in $\phi$ and its derivatives, so that the equation takes the form

\[
a(x, y)\phi_{xx} + 2b(x, y)\phi_{xy} + c(x, y)\phi_{yy} + d(x, y)\phi_x + e(x, y)\phi_y + f(x, y)\phi + g(x, y) = 0 \tag{7}
\]
then the equation is called a “linear PDE”. Otherwise, if the linearity is restricted to the second order derivatives, the equation is called a “semilinear PDE” or “quasilinear PDE.” As simple examples, the equation of Klein-Gordon

\[ \phi_{tt}(x, t) - \phi_{xx}(x, t) + \phi(x, t) = 0 \]  

(8)

is a linear PDE, while the equation of sine-Gordon

\[ \phi_{tt}(x, t) - \phi_{xx}(x, t) + \sin \phi(x, t) = 0 \]  

(9)

is a semilinear PDE. They both describe the propagation of waves. (Dimensionful constants have been omitted for simplicity.)

It is very important to remember that a PDE is not well defined without specification of the boundary conditions. Similarly, the differential operators which appear in the PDEs require the specification of boundary conditions for their definition. Looking at the simple example of one-dimensional systems, the second derivative operator

\[ D_2 = \frac{d^2}{dx^2} \]  

(10)

acting on functions \( \phi(x) \) defined over \( 0 \leq x \leq L \) with boundary conditions \( \phi(0) = \phi(L) = 0 \) is a different operator from

\[ \tilde{D}_2 = \frac{d^2}{dx^2} \]  

(11)

acting on functions \( \phi(x) \) defined over \( -\infty < x < \infty \) which decrease fast enough for \( x \to \pm \infty \) to insure that \( \int_{-\infty}^{\infty} \phi(x)^2 \, dx \) is finite. \( D_2 \) has a discrete spectrum of normalizable eigenfunctions. \( \tilde{D}_2 \) has a continuous spectrum of non-normalizable eigenfunctions.

Insofar as boundary conditions (bc) are concerned, it is good to keep in mind a few of the most common ones, with the associated terminology:

- **Dirichlet bc** - the function \( \phi \) takes a definite value, often zero, at the boundary.
- **Neumann bc** - the derivative of the function \( \phi \) takes a definite value, often zero, at the boundary.
- **Periodic bc** - the function is defined over a domain \( 0 \leq x \leq L \) and the value taken by \( \phi \) and \( \phi_x \) are the same at \( x = 0 \) and \( x = L \). The domain of definition of \( \phi \) has the topology of a circle, or of a cylinder if in more dimensions. It could also have the topology of a torus if \( \phi \) satisfies periodic bc over two or more variables.
- **Cauchy bc** (or Cauchy data) - the function \( \phi \) takes a definite value over a given line or surface \( \gamma \) and the derivative \( \phi_n \) of \( \phi \) in the direction normal to \( \gamma \) is also specified.
But the system could also satisfy mixed boundary conditions, for example \( \phi(0) = 0 \) (Dirichlet bc) and \( \phi_x(L) = 0 \) (Neumann bc) at the ends of its domain of definition, or other types of boundary conditions, such as \( \phi_x(0) = \alpha \phi(0) \) where \( \alpha \) is a given constant.

**Broad classification of PDEs.**

PDEs may describe the evolution of a system in time. Such is the case, for example, of the D’Alembert equation, or wave equation, which in three dimensions reads

\[
\frac{\partial^2 \phi(x, y, z, t)}{\partial t^2} - v^2 \Delta \phi(x, y, z, t) = 0 \quad (12)
\]

where \( \Delta \) stands for the Laplacian operator and the velocity \( v \) may be a constant or depend on the \( x, y, z \) coordinates.

It is natural to refer to equations of this type as “evolution PDEs.”

Equations 8, 9 are other examples of evolution PDEs, as is the time dependent Schrödinger equation

\[
\hat{h} \frac{\partial \psi(x, t)}{\partial t} = -\frac{\hbar^2}{2m} \frac{\partial^2 \psi(x, t)}{\partial x^2} + V(x) \psi(x, t) \quad (13)
\]

in one space dimension, or

\[
\hat{h} \frac{\partial \psi(x, y, z, t)}{\partial t} = -\frac{\hbar^2}{2m} \Delta \psi(x, y, z, t) + V(x) \psi(x, y, z, t) \quad (14)
\]

in a three dimensional setup.

Other PDEs serve instead to define the overall shape of a physical quantity, such as an electric or magnetic potential as determined by an external agent, for example a distribution of charges or currents, or by the demand that the quantity be an eigenfunctions of some definite operator, as in the case of the time independent Schrödinger wave function. A few examples of such equations are the Poisson equation for the electrostatic potential \( \phi \)

\[
\Delta \psi(x, y, z) = -\frac{\rho(x, y, z)}{\epsilon} \quad (15)
\]

where \( \rho(x, y, z) \) is the charge density and \( \epsilon \) the permittivity of the medium, or the Laplace equation for the electrostatic potential \( \phi \) in absence of charges

\[
\Delta \psi(x, y, z) = 0 \quad (16)
\]

where, however, \( \phi \) is known to take definite values at the boundary of the system (for example on some conducting surfaces.)

A further example is the Schrödinger equation for the energy eigenfunction corresponding to a certain energy eigenvalue \( E \)

\[
-\frac{\hbar^2}{2m} \Delta \psi(x, y, z) + V(x) \psi(x, y, z) = E \psi(x, y, z) \quad (17)
\]
It is less obvious, as opposed to the evolution PDEs, how one should refer to these equations. We may call them “conformation” PDEs, in agreement with the definition from dictionary.com: “conformation, noun, = manner of formation; structure; form, as of a physical entity.” But we will not have much use for this terminology. It will be sufficient to keep in mind the clear distinction between these equations and the evolution PDEs.

The Cauchy problem for a quasilinear PDE.


Consider the quasilinear PDE in two dimensions Eq. 6, which we reproduce here for convenience

\[ a(x,y) \phi_{xx} + 2b(x,y) \phi_{xy} + c(x,y) \phi_{yy} + r(\phi_x, \phi_y, \phi, x, y) = 0 \quad \text{Eq. (6)} \]

Let \( \gamma \) be a curve in the \( x,y \)-plane parametrically described by

\[
\begin{align*}
    x &= u(s) \\
    y &= v(s)
\end{align*}
\]  

(18)  

(19)  

The unit normal to the curve \( \gamma \) will have components

\[ \hat{n} = \frac{(-v'(s), u'(s))}{\sqrt{u'(s)^2 + v'(s)^2}} \]  

(20)  

The Cauchy problem consists in determining the solution to the PDE in the neighborhood of \( \gamma \), given the values on \( \gamma \) taken by \( \phi \) and by its derivative \( \phi_n \) along the normal to the curve. So, we assume that

\[ \phi(s) = \phi(u(s), v(s)) \]  

(21)  

is a given function, which we take as part of our Cauchy boundary conditions. Insofar as \( \phi_n(s) \) is concerned, if \( \phi_x(x,y), \phi_y(x,y) \) are the first order partial derivatives of the solution to the PDE which we are planning to find, then \( \phi_n(s) \) is given by

\[ \phi_n(s) = \frac{-\phi_x(u(s), v(s))v'(s) + \phi_y(u(s), v(s))u'(s)}{\sqrt{u'(s)^2 + v'(s)^2}} \]  

(22)  

Notice that the values \( \phi_x(u(s), v(s)), \phi_y(u(s), v(s)) \) of the partial derivatives of \( \phi \) on \( \gamma \) are not yet known at this stage. We only demand that the normal derivative takes a given value \( \phi_n(s) \) on \( \gamma \), as the other part of the Cauchy boundary conditions. The idea now is to solve the Cauchy problem by finding all the partial derivatives of \( \phi \) on any point of gamma, \( x_0 = u(s_0), y_0 = v(s_0) \), through the Cauchy boundary conditions and the differential equation 6 satisfied by \( \phi(x,y) \). We will illustrate here the procedure up to the determination of all partial derivatives up to the second order, but the procedure could in principle be continued through the determination of the derivatives of any order.
Since we know $\phi(s)$ on $\gamma$ we may calculate $\phi'(s)$ from the given Cauchy boundary conditions. On the other hand $\phi'(s)$ is given by

$$\phi'(s) = \phi_x(u(s), v(s))u'(s) + \phi_y(u(s), v(s))v'(s)$$  \hspace{1cm} (23)

Equations 23, 22 give us then

$$\phi_x(x_0, y_0)u'(s_0) + \phi_y(x_0, y_0)v'(s_0) = \phi'(s_0)$$  \hspace{1cm} (24)
$$\frac{-\phi_x(x_0, y_0)v'(s_0) + \phi_y(x_0, y_0)u'(s_0)}{\sqrt{u'(s_0)^2 + v'(s_0)^2}} = \phi_n(s_0)$$  \hspace{1cm} (25)

which is a system of two linear equations in $\phi_x(x_0, y_0)$ and $\phi_y(x_0, y_0)$ whose determinant $\sqrt{u'(s_0)^2 + v'(s_0)^2}$ cannot vanish for a non-singular parametrization of $\gamma$. So we can solve the equations and, at this point, we will know $\phi(x_0, y_0)$, $\phi_x(x_0, y_0)$, $\phi_y(x_0, y_0)$.

We turn now to the determination of the second order derivatives. Since $\phi_x(x, y), \phi_y(x, y)$ can be calculated not only at $x_0, y_0$ but for points on $\gamma$ in the neighborhood of $x_0, y_0$, we will be able to calculate $d\phi_x(u(s), v(s))/ds$ and $d\phi_y(u(s), v(s))/ds$ at $s = s_0$. On the other hand we have

$$\left.\frac{d\phi_x(u(s), v(s))}{ds}\right|_{s=s_0} = \phi_{xx}(x_0, y_0)u'(s_0) + \phi_{xy}(x_0, y_0)v'(s_0)$$  \hspace{1cm} (26)
$$\left.\frac{d\phi_y(u(s), v(s))}{ds}\right|_{s=s_0} = \phi_{yx}(x_0, y_0)u'(s_0) + \phi_{yy}(x_0, y_0)v'(s_0)$$  \hspace{1cm} (27)

where, again, $\phi_{xx}(x_0, y_0), \phi_{xy}(x_0, y_0), \phi_{yy}(x_0, y_0)$ are not known yet, but are the second order derivatives we wish to determine.

We complement Eqs. 26, 27 with the differential equation satisfied by $\phi$ at $(x_0, y_0)$:

$$a(x_0, y_0)\phi_{xx}(x_0, y_0) + 2b(x_0, y_0)\phi_{xy}(x_0, y_0) + c(x_0, y_0)\phi_{yy}(x_0, y_0)$$
$$+ r(\phi_x(x_0, y_0), \phi_y(x_0, y_0), \phi(x_0, y_0), x_0, y_0) = 0$$  \hspace{1cm} (28)

The three equations

$$\phi_{xx}(x_0, y_0)u'(s_0) + \phi_{xy}(x_0, y_0)v'(s_0) = \left.\frac{d\phi_x(u(s), v(s))}{ds}\right|_{s=s_0}$$  \hspace{1cm} (29)
$$\phi_{xy}(x_0, y_0)u'(s_0) + \phi_{yy}(x_0, y_0)v'(s_0) = \left.\frac{d\phi_y(u(s), v(s))}{ds}\right|_{s=s_0}$$  \hspace{1cm} (30)
$$a(x_0, y_0)\phi_{xx}(x_0, y_0) + 2b(x_0, y_0)\phi_{xy}(x_0, y_0) + c(x_0, y_0)\phi_{yy}(x_0, y_0)$$
$$- r(\phi_x(x_0, y_0), \phi_y(x_0, y_0), \phi(x_0, y_0), x_0, y_0) = 0$$  \hspace{1cm} (31)

form a system of three linear equations in the three unknowns $\phi_{xx}(x_0, y_0), \phi_{xy}(x_0, y_0), \phi_{yy}(x_0, y_0)$. For clarity, let us write the equations in matrix form

$$\begin{pmatrix}
  u'(s_0) & v'(s_0) & 0 \\
  0 & u'(s_0) & v'(s_0) \\
  a(x_0, y_0) & 2b(x_0, y_0) & c(x_0, y_0)
\end{pmatrix}
\begin{pmatrix}
  \phi_{xx}(x_0, y_0) \\
  \phi_{xy}(x_0, y_0) \\
  \phi_{yy}(x_0, y_0)
\end{pmatrix}
= \begin{pmatrix}
  \left.\frac{d\phi_x(u(s), v(s))}{ds}\right|_{s=s_0} \\
  \left.\frac{d\phi_y(u(s), v(s))}{ds}\right|_{s=s_0} \\
  - r(\phi_x, \phi_y, \phi, x_0, y_0)
\end{pmatrix}$$  \hspace{1cm} (32)
These equations will have a solution for \( \phi_{xx}(x_0, y_0), \phi_{xy}(x_0, y_0), \phi_{yy}(x_0, y_0) \) provided that the determinant \( D \) of the matrix of coefficients

\[
D = c(x_0, y_0)u'(s_0)^2 + a(x_0, y_0)v'(s_0)^2 - 2b(x_0, y_0)u'(x_0)v'(x_0)
\]  

(33)
does not vanish. Then one can determine all the second derivatives at \( x_0, y_0 \) and the construction, although cumbersome, can be continued through the determination of all partial derivatives of the function \( \phi(x, y) \) at \( x_0, y_0 \). With the value of \( \phi \) and of all its partial derivatives in hand, \( \phi(x, y) \) can be found in a neighborhood of \( x_0, y_0 \) by a double Taylor series expansion. This procedure is at the root of the Cauchy-Kowalewski theorem of the local existence and uniqueness of a solution for second order quasilinear PDEs in two independent variables.

The procedure will fail, however, if the determinant \( D \) in Eq. 33 vanishes. In this case it will not be possible to solve Eqs. 29-31 for the second derivatives, which means in turn that the Cauchy boundary conditions are inconsistent. The value of \( D \), which we should more properly denote by \( D(\gamma, x_0, y_0) \), will depend on the chosen curve \( \gamma \) and the point \( x_0, y_0 \) on it. The equation

\[
D(\gamma, x_0, y_0) = c(x_0, y_0)u'(s_0)^2 + a(x_0, y_0)v'(s_0)^2 - 2b(x_0, y_0)u'(x_0)v'(x_0) = 0
\]  

(34)
is solved by

\[
\frac{v'(s_0)}{u'(s_0)} = \frac{b(x_0, y_0) \pm \sqrt{b(x_0, y_0)^2 - a(x_0, y_0)c(x_0, y_0)}}{a(x_0, y_0)}
\]  

(35)
There are then three possibilities:

- \( a(x_0, y_0)c(x_0, y_0) > b(x_0, y_0)^2 \). There are then no real solutions to Eq. 35 and thus no curves \( \gamma \) passing through \( x_0, y_0 \) for which the Cauchy-Kowalewski construction cannot be carried out. In the case the equation is called “elliptic” at the point \( x_0, y_0 \).

- \( a(x_0, y_0)c(x_0, y_0) = b(x_0, y_0)^2 \). There is then one real solution to Eq. 35. For most equations the equality will hold true in a neighborhood of \( x_0, y_0 \). Then there will be one curve \( \gamma \) passing through \( x_0, y_0 \) for which the Cauchy-Kowalewski construction cannot be carried out. In the case the equation is called “parabolic” at the point \( x_0, y_0 \).

- \( a(x_0, y_0)c(x_0, y_0) < b(x_0, y_0)^2 \). By continuity this inequality will hold true in a neighborhood of \( x_0, y_0 \). There will be then two real solutions to Eq. 35 and it will be possible to integrate the equations to find two curves \( \gamma \) passing through \( x_0, y_0 \) for which the Cauchy-Kowalewski construction cannot be carried out. In the case the equation is called “hyperbolic” at the point \( x_0, y_0 \) and the two curves \( \gamma \) are called the “characteristics” of the hyperbolic equation.

For the major part of the PDEs one encounters in physics or engineering, the type of the equation, i.e. elliptic, parabolic, or hyperbolic, will be constant throughout its domain of
definition, however there are cases where this is not true. A notable example is the “Tricomi” equation\(^1\)

\[
\phi_{yy} - y\phi_{xx} = 0 \tag{36}
\]

With \(a = -y, b = 0, c = 1\), we will have \(ac < b^2\) for \(y > 0\), \(ac = b^2\) for \(y = 0\), \(ac > b^2\) for \(y < 0\), so the Tricomi equation is hyperbolic in the upper half plane, parabolic on the \(x\) axis, elliptic in the lower half plane.

**Characteristics of a hyperbolic PDE.**

The characteristics of a hyperbolic PDE are the curves \(x = u(s), y = v(s)\) which satisfy the equation

\[
\frac{v'(s)}{u'(s)} = \frac{b(x, y) \pm \sqrt{b(x, y)^2 - a(x, y)c(x, y)}}{a(x, y)} \tag{37}
\]

By eliminating \(ds\) we obtain the equation

\[
\frac{dy}{dx} = \frac{b(x, y) \pm \sqrt{b(x, y)^2 - a(x, y)c(x, y)}}{a(x, y)} \tag{38}
\]

which can be solved to find \(y = y(x)\). Notice that, since we have assumed that the coefficients \(a, b, c\) are functions of position only, i.e. that they do not have an implicit dependence on the solution, the characteristics are independent of the solution and are fully determined by the coefficients of the second order derivatives. As simple examples of PDEs of the three different types, the Laplace equation

\[
\frac{\partial^2 \phi(x, y)}{\partial x^2} + \frac{\partial^2 \phi(x, y)}{\partial y^2} = 0 \tag{39}
\]

with \(a = c = 1\) is an elliptic equation; the “heat diffusion equation”

\[
\frac{\partial \phi(x, t)}{\partial t} - D\frac{\partial^2 \phi(x, y)}{\partial x^2} = 0 \tag{40}
\]

\((D\) denotes here the diffusion constant,\) with \(a = b = 0, c = -D\), is a parabolic equation; the wave-equation

\[
\frac{\partial^2 \phi(x, t)}{\partial t^2} - v^2\frac{\partial^2 \phi(x, y)}{\partial x^2} = 0 \tag{41}
\]

\((v\) denotes here the velocity of propagation of the waves,\) with \(a = 1, b = 0, c = -v^2\), is a hyperbolic equation\(^2\). Its characteristics satisfy

\[
\frac{dt}{dx} = \pm \frac{1}{v} \tag{42}
\]

\(^1\)From the mathematician Francesco Tricomi, who was one of my professors at the University of Torino (C.R.)

\(^2\)In Equationa 40 and 41 we changed the independent variables from \(x, y\) to \(t, x\) to give the equations their habitual form.
and are the lines 

\[ x \pm vt = \text{constant} \]  

(43)

**Characteristics as lines of propagation of discontinuities.**

The characteristics of a PDE are also the lines of propagation of discontinuities. Let us imagine that a line \( \gamma \) divides the \( x - y \) into two regions \( R_1 \) and \( R_2 \), where the solution of the hyperbolic PDE takes values \( \phi^{(1)}(x, y) \) and \( \phi^{(2)}(x, y) \). We will demand that the solution and its first order derivatives be continuous across \( \gamma \), but allow for a discontinuity in the second order derivatives of \( \phi \). It is useful to establish some notation. Let the equation of \( \gamma \) be given by

\[ x = f(y) \]  

(44)

Let us also denote by square brackets the discontinuity, or “jump”, of any function \( h(x, y) \) across the divide \( \gamma \) between \( R_1 \) and \( R_2 \):

\[ [h](f(y), y) = h^{(2)}(f(y), y) - h^{(1)}(f(y), y) \]  

(45)

where, with obvious notation, \( h^{(1)}(f(y), x) \) and \( h^{(2)}(f(y), x) \) represent the values taken by \( h \) on \( \gamma \) if coming from region \( R_1 \) or, respectively, from region \( R_2 \). So we are assuming that

\[ [\phi](f(y), y) = [\phi_x](f(y), y) = [\phi_y](f(y), y) = 0 \]  

(46)

but allow for a finite jump of the second derivatives.

We note that the jumps \( [\phi_{xx}](f(y), y), [\phi_{xy}](f(y), y), [\phi_{yy}](f(y), y) \) are related. Let us denote the jump of \( \phi_{xx} \) by \( \lambda(y) \). We will then have

\[ [\phi_{xx}](f(y), y) = \lambda(y) \]  

(47)

\[ [\phi_{xy}](f(y), y) = -\lambda(y)f'(y) \]  

(48)

\[ [\phi_{yy}](f(y), y) = \lambda(y)f'(y)^2 \]  

(49)

Indeed we have

\[ 0 = [\phi_x](f(y), y) = \phi^{(2)}_x(f(y), y) - \phi^{(1)}_x(f(y), y) \]  

(50)

\[ 0 = [\phi_y](f(y), y) = \phi^{(2)}_y(f(y), y) - \phi^{(1)}_y(f(y), y) \]  

(51)

By taking the derivative of these equations with respect to \( y \) we get

\[ \phi^{(2)}_{xx}(f(y), y)f'(y) - \phi^{(1)}_{xx}(f(y), y)f'(y) + \phi^{(2)}_{xy}(f(y), y)f'(y) - \phi^{(1)}_{xy}(f(y), y) = 0 \]  

(52)

\[ \phi^{(2)}_{xy}(f(y), y)f'(y) - \phi^{(1)}_{xy}(f(y), y)f'(y) + \phi^{(2)}_{yy}(f(y), y)f'(y) - \phi^{(1)}_{yy}(f(y), y) = 0 \]  

(53)

or

\[ [\phi_{xx}](f(y), y)f'(y) + [\phi_{xy}](f(y), y) = 0 \]  

(54)

\[ [\phi_{xy}](f(y), y)f'(y) + [\phi_{yy}](f(y), y) = 0 \]  

(55)
from which Eqs. 48, 49 immediately follow.

Now, since \( \phi \) satisfies the differential equation

\[
a(x, y)\phi_{xx} + 2b(x, y)\phi_{xy} + c(x, y)\phi_{yy} + r(\phi_x, \phi_y, \phi, x, y) = 0
\]

in both regions and \( \phi, \phi_x, \phi_y \) are continuous across \( \gamma \), it follows that the jumps in the second derivatives will satisfy the equation

\[
a(x, y)[\phi_{xx}](f(y), y) + 2b(x, y)[\phi_{xy}](f(y), y) + c(x, y)[\phi_{yy}](f(y), y) = 0
\] (56)

By virtue of Eqs. 47, 48, 49 this equation can be written as

\[
a(x, y)\lambda(y) - 2b(x, y)\lambda(y) f'(y) + c(x, y)\lambda(y)f'(y)^2 = 0
\] (57)

or, factoring out \( \lambda(y) \)

\[
a(x, y) - 2b(x, y) \frac{dx}{dy} + c(x, y) \left[ \frac{dx}{dy} \right]^2 = 0
\] (58)

But this is the equation satisfied by the characteristics of the PDE (see Eqs. 34 and 38.) We conclude that the discontinuity in the second derivatives propagates along the characteristics of the equation. If the PDE is linear (as opposed to quasilinear) is also possible to derive a first order differential equation for the way the jump \( \lambda(y) \) propagates along \( \gamma \). We will not pursue this here, but the equation shows that if \( \lambda(y) = 0 \), i.e. if the second derivatives are continuous at some point across a characteristic line, they will be continuous all along the line, as one might intuitively expect.

**Canonical form of linear PDEs**

Imagine we change coordinates from \( x, y \) to \( \xi, \eta \) with \( x = x(\xi, \eta), y = y(\xi, \eta) \) and \( \xi = \xi(x, y), \eta = \eta(x, y) \). The function \( \phi(x, y) \) will be mapped to a new function \( \psi(\xi, \eta) \) with

\[
\psi(\xi, \eta) = \phi(x(\xi, \eta), y(\xi, \eta))
\] (59)

Let us focus on the term with the second order derivatives in the PDE

\[
a(x, y) \frac{\partial^2 \phi(x, y)}{\partial x^2} + 2b(x, y) \frac{\partial^2 \phi(x, y)}{\partial x \partial y} + c(x, y) \frac{\partial^2 \phi(x, y)}{\partial y^2} + \cdots = 0
\] (60)

where the dots stand for terms linear in \( \phi_x, \phi_y \) and \( \phi \). We have

\[
\frac{\partial}{\partial x} = \frac{\partial \xi}{\partial x} \frac{\partial}{\partial \xi} + \frac{\partial \eta}{\partial x} \frac{\partial}{\partial \eta}
\]

(61)

\[
\frac{\partial}{\partial y} = \frac{\partial \xi}{\partial y} \frac{\partial}{\partial \xi} + \frac{\partial \eta}{\partial y} \frac{\partial}{\partial \eta}
\]

(62)

Substituting these relations into Eq. 60 we obtain an equation for \( \psi \)

\[
A(\xi, \eta) \frac{\partial^2 \psi(\xi, \eta)}{\partial \xi^2} + 2B(\xi, \eta) \frac{\partial^2 \psi(\xi, \eta)}{\partial \xi \partial \eta} + C(\xi, \eta) \frac{\partial^2 \psi(\xi, \eta)}{\partial \eta^2} + \cdots = 0
\] (63)
with

\[ A(\xi, \eta) = \left[ \frac{\partial \xi}{\partial x} \right]^2 a(x, y) + 2 \frac{\partial \xi}{\partial x} \frac{\partial \eta}{\partial y} b(x, y) + \left[ \frac{\partial \eta}{\partial y} \right]^2 c(x, y) \]  
\[ B(\xi, \eta) = \frac{\partial \xi}{\partial x} \frac{\partial \eta}{\partial x} a(x, y) + \left[ \frac{\partial \xi}{\partial x} \frac{\partial \eta}{\partial y} + \frac{\partial \xi}{\partial y} \frac{\partial \eta}{\partial x} \right] b(x, y) + \frac{\partial \xi}{\partial x} \frac{\partial \eta}{\partial y} c(x, y) \]  
\[ C(\xi, \eta) = \left[ \frac{\partial \eta}{\partial x} \right]^2 a(x, y) + 2 \frac{\partial \eta}{\partial x} \frac{\partial \eta}{\partial y} b(x, y) + \left[ \frac{\partial \eta}{\partial y} \right]^2 c(x, y) \]

We will consider in detail only the case of a hyperbolic PDE. In this case it is possible to use the change of coordinates to eliminate the terms with the double derivatives, i.e. to make the coefficients \( A(\xi, \eta) \) and \( C(\xi, \eta) \) both equal to zero. To accomplish this we need to find \( \xi \) and \( \eta \) that satisfy the equations

\[ a(x, y)[\xi_x(x, y)^2 + 2b(x, y)\xi_x(x, y)\xi_y(x, y) + c(x, y)[\xi_y(x, y)]^2 = 0 \]  
\[ a(x, y)[\eta_x(x, y)^2 + 2b(x, y)\eta_x(x, y)\eta_y(x, y) + c(x, y)[\eta_y(x, y)]^2 = 0 \]

The two equations are identical and can be solved by taking advantage of the characteristics. The characteristic lines satisfy Eq. 38. Let us consider the family of solutions with the positive sign in the equation, solving

\[ \frac{dy}{dx} = \frac{b(x, y) + \sqrt{b(x, y)^2 - a(x, y)c(x, y)}}{a(x, y)} \]

and let us represent them in implicit form by an equation

\[ f(x, y) = 0 \]

For example, if Eq. 69 is solved by \( y = g(x) \) the function \( f(x, y) \) could simply be \( f(x, y) = y - g(x) \). However, there will be (infinitely) many characteristic lines that solve Eq. 69. So the function \( f(x, y) \) must also depend on a parameter that specifies a particular solution. As we will presently show, we can take this parameter to be the new coordinate \( \xi \). Thus \( \xi \) will be implicitly defined by

\[ f(x, y, \xi) = 0 \]

For example, the characteristics of the wave-equation

\[ \frac{\partial^2 \phi(x, t)}{\partial t^2} - v^2 \frac{\partial^2 \phi(x, t)}{\partial x^2} = 0 \]

are the lines \( x \pm vt = \text{constant} \). So we may take the characteristics with the positive sign and parametrize them by

\[ f(x, y, \xi) \equiv x + vt - \xi = 0 \]

This will define the \( \xi \) coordinate as \( x + vt \). Let us show now that the coordinate defined in the manner described above satisfies Eq. 67. As a function of \( x, y \) the coordinate \( \xi \) will satisfy the equation

\[ f(x, y, \xi(x, y)) = 0 \]
By taking the derivatives of this equation by respect to \( x \) or \( y \) we get

\[
\frac{\partial f(x, y, \xi)}{\partial x} + \frac{\partial f(x, y, \xi)}{\partial \xi} \xi_x = 0 
\]

(75)

\[
\frac{\partial f(x, y, \xi)}{\partial y} + \frac{\partial f(x, y, \xi)}{\partial \xi} \xi_y = 0
\]

(76)

On the other hand, if we keep \( \xi \) fixed, the equation \( f(x, y, \xi) = 0 \) is the equation of one of the characteristic lines. By taking the differential of this equation we find

\[
\frac{\partial f(x, y, \xi)}{\partial x} \, dx + \frac{\partial f(x, y, \xi)}{\partial y} \, dy = 0
\]

(77)

By substituting \( dy \) from Eq. 69 and eliminating the common factor \( dx \) we obtain

\[
\frac{\partial f(x, y, \xi)}{\partial x} + \frac{b(x, y) + \sqrt{b(x, y)^2 - a(x, y)c(x, y)}}{a(x, y)} \frac{\partial f(x, y, \xi)}{\partial y} = 0
\]

(78)

which, after some simple algebra, gives

\[
a(x, y) \left[ \frac{\partial f(x, y, \xi)}{\partial x} \right]^2 + 2b(x, y) \frac{\partial f(x, y, \xi)}{\partial x} \frac{\partial f(x, y, \xi)}{\partial y} + \left[ \frac{\partial f(x, y, \xi)}{\partial y} \right]^2 = 0
\]

(79)

Finally, by using Eqs. 75, 76 to replace \( \partial f/\partial x \) and \( \partial f/\partial y \) with \( (\partial f/\partial \xi) \xi_x, (\partial f/\partial \xi) \xi_y \), and factoring out \( \partial f/\partial \xi \), which will not vanish if \( \xi \) parametrizes different characteristic lines, we obtain

\[
a(x, y) [\xi_x(x, y)]^2 + 2b(x, y) \xi_x(x, y) \xi_y(x, y) + c(x, y)[\xi_y(x, y)]^2 = 0
\]

i.e. we see that \( \xi(x, y) \) satisfies Eq.67. The other characteristic lines, those obtained with the negative sign in front of the square root in Eq. 38, can be similarly used to define the new coordinate \( \eta \) as function of \( x \) and \( y \). What emerges from this construction is that if we use the two sets of characteristic lines as the new coordinate lines, the differential equation will take the form

\[
2B(\xi, \eta) \frac{\partial^2 \psi(\xi, \eta)}{\partial \xi \partial \eta} + R(\phi_\xi, \phi_\eta, \phi, \xi, \eta) = 0
\]

(80)

or

\[
\frac{\partial^2 \psi(\xi, \eta)}{\partial \xi \partial \eta} + S(\phi_\xi, \phi_\eta, \phi, \xi, \eta) = 0
\]

(81)

with \( S = R/(2B) \), where \( R \) or \( S \) are linear functions of \( \phi_\xi, \phi_\eta \) and \( \phi \). If we take as new variables \( \tilde{x} = \xi - \eta, \tilde{y} = \xi + \eta \), the equation will take the form

\[
\frac{\partial^2 \psi(\tilde{x}, \tilde{y})}{\partial \tilde{y}^2} - \frac{\partial^2 \psi(\tilde{x}, \tilde{y})}{\partial \tilde{x}^2} + D(\tilde{x}, \tilde{y})\psi_{\tilde{x}} + E(\tilde{x}, \tilde{y})\psi_{\tilde{y}} + F(\tilde{x}, \tilde{y})\psi + G(\tilde{x}, \tilde{y}) = 0
\]

(82)

which is the canonical form of a hyperbolic linear PDE in two dimensions.
It is also possible to show that, by a suitable change of coordinates, an elliptic linear PDE can be brought to the canonical form

$$\frac{\partial^2 \psi(\tilde{x}, \tilde{y})}{\partial \tilde{x}^2} + \frac{\partial^2 \psi(\tilde{x}, \tilde{y})}{\partial \tilde{y}^2} + D(\tilde{x}, \tilde{y})\psi_2 + E(\tilde{x}, \tilde{y})\psi_y + F(\tilde{x}, \tilde{y})\psi + G(\tilde{x}, \tilde{y}) = 0 \quad (83)$$

and a parabolic linear PDE to the canonical form

$$\frac{\partial^2 \psi(\tilde{x}, \tilde{y})}{\partial \tilde{x}^2} + D(\tilde{x}, \tilde{y})\psi_2 + E(\tilde{x}, \tilde{y})\psi_y + F(\tilde{x}, \tilde{y})\psi + G(\tilde{x}, \tilde{y}) = 0 \quad (84)$$

**PDEs in more than two dimensions.**

Many of the considerations developed above carry over to systems of higher dimensionality. Let us denote the independent variables, or coordinates, by $x_i$ with $i = 1 \ldots d$, $d$ being the dimensionality of the space, or space-time, underlying the physical system, and let us denote the dependent variable by $\phi(x_i)$. The general form of a linear or semilinear second order PDE will be

$$\sum_{j_1,j_2} A_{j_1,j_2}(x_i) \frac{\partial^2 \phi(x_i)}{\partial x_{j_1} \partial x_{j_2}} + R(\phi_{x_i}, \phi, x_i) = 0 \quad (85)$$

where, as we have done up to now, we assume that the matrix $A_{j_1,j_2}(x_i)$ only depends on the coordinates. If the function $R(\phi_{x_i}, \phi, x_i)$ is linear in the first order derivatives $\phi_{x_i}$ and $\phi$ then the equation is linear, otherwise the equation is semilinear. The type of the equation will depend on the eigenvalues of the $d \times d$ matrix $A$:

- If the $d$ eigenvalues are non-zero and all of the same sign, the equation will be an elliptic PDE. Typically it will be what at the very beginning of these notes we called a conformation equation.

- If one of the eigenvalues is zero and the others are non-zero and all of the same sign, the equation will be a parabolic PDE. Typically the equation will describe the evolution of the system in some variable $t$ and will contain only the first order derivative of $\phi$ with respect to $t$, and the second order derivatives with respect to the other independent variables.

- If all the eigenvalues are non-zero and $d - 1$ of them have the same sign while the remaining eigenvalue will be of the opposite sign, then the equation will be a hyperbolic PDE and will be an evolution equation.

- Four dimensional equations where $A$ has two positive and two negative non-zero eigenvalues are not likely to be encountered.

The Cauchy problem in $d > 2$ dimensions consists in assigning the value of $\phi$ on a smooth, continuous manifold $\Gamma$ of dimension $d - 1$ (a surface if $d = 3$, a volume if $d = 4$) as well as the
derivative of \( \phi \) in the direction normal to the manifold, and using these data to reconstruct the solution in a neighborhood of \( \Gamma \). This will always be possible with elliptic PDEs, but in the case of hyperbolic PDEs there will be manifolds \( \Gamma \) for which assigning the Cauchy data leads to an inconsistency. These manifold are the characteristic surfaces or hypersurfaces of the PDE. If the PDE describes the evolution of a system, with two or three space variables \( \vec{x} \), and an evolution coordinate \( t \), and if we consider a point-like perturbation at \( \vec{x} = \vec{x}_0 \) occurring at \( t = t_0 \), the propagation in \( t \) of the perturbation will be bound by the envelope of characteristics passing through \( \vec{x}_0, t_0 \). If the equation describes the propagation of light in the vacuum, this envelope will be the “light-cone” emanating from \( \vec{x}_0, t_0 \).