General considerations.

We briefly address the problem of nonlinear differential equations. Since we will be dealing with evolution equations, we will use here the notation $t$ for the independent variable, and $x(t), y(t)$, or $x_i(t)$ for the dependent variables.

It is useful to note that higher order ODEs can be converted to systems of differential equations of the first order. Consider, for example, the equation

$$x''(t) = F(x'(t), x(t), t)$$  \hspace{1cm} (1)

Let us introduce $y(t) = x'(t)$ as a separate function. Then Eq. 1 can be recast as a system of first order equations:

$$x'(t) = y(t)$$  \hspace{1cm} (2)
$$y'(t) = F(y(t), x(t), t)$$  \hspace{1cm} (3)

If $F$ is a linear function of $y(t)$ and $x(t)$, e.g.

$$F = -P(t)y(t) - Q(t)x(t) + R(t)$$  \hspace{1cm} (4)

then we fall back to the linear ODEs which have been considered in the notes on linear ODEs. Otherwise the equation is nonlinear.

Systems of first order nonlinear ODEs are more general than the one in Eqs. 2 and 3. They can be of the form

$$x'(t) \equiv \frac{dx}{dt} = G(y(t), x(t), t)$$  \hspace{1cm} (5)
$$y'(t) \equiv \frac{dy}{dt} = F(y(t), x(t), t)$$  \hspace{1cm} (6)

If Eq. 5 can be solved to give

$$y(t) = H(x'(t), x(t), t)$$  \hspace{1cm} (7)

then, by differentiating, we get

$$y'(t) = \frac{\partial H(x', x, t)}{\partial x'} x'' + \frac{\partial H(x', x, t)}{\partial x} x' + \frac{\partial H(x', x, t)}{\partial t}$$  \hspace{1cm} (8)

and substituting into Eq. 6 we obtain

$$\frac{\partial H(x', x, t)}{\partial x'} x'' + \frac{\partial H(x', x, t)}{\partial x} x' + \frac{\partial H(x', x, x)}{\partial t} - F(H(x'(t), x(t), y)) = 0$$  \hspace{1cm} (9)
which is an equation involving only \(x, x', x''\), but the procedure may not be feasible or yield manageable results, in which case one should stay with the system of coupled nonlinear, first order ODEs.

The system of equations 5, 6 can be generalized to the situation where we have more than two dependent variables \(x_1, x_2, \ldots, x_n\), in which case it will take the form

\[
\frac{dx_i}{dt} = F_i(x_j(t), t) \quad i = 1, 2, \ldots, n
\]  

(10)

The solutions of nonlinear ODE can exhibit quite a wide variety of behaviors and, depending on the equations, the results which can be established analytically may be limited. Quite often the equations need to be treated by computational techniques. A very general result which can be established analytically, however, is that, in a region where the functions \(F_i\) and their derivatives are well defined, a set of initial values

\[x_i(t_0) = x_{i,0}\]  

(11)

determines the solution \(x_i(t)\) uniquely. As a consequence different trajectories, i.e. different solutions to the equations, cannot have any point in common (for a given value of \(t\)). If \(x_i(t)\) and \(\tilde{x}_i(t)\) were to coincide at some \(t = t_1\), then using \(t_1\) as the initial value in the evolution equations, with initial data \(x_i(t_1) = \tilde{x}_i(t_1)\), we would have \(x_i(t) = \tilde{x}_i(t)\) for all \(t\).

The space spanned by the variables \(x_i(t)\) is often referred to as the phase space. The trajectories \(x_i(t)\) span curves in phase space. Let \(\gamma\) be the curve spanned by the trajectory with initial data \(x_i(t_0) = x_{i,0}\), and \(\gamma'\) be the curve spanned by the trajectory with initial data \(x_i(t_1) = x_{i,0}\) (note the different value of the initial time, with identical initial value of the dependent variables.) If the functions \(F_i(x_j(t), t)\) in Eq. 10 have an explicit time dependence, then \(\gamma\) and \(\gamma'\) can be different. To understand this point, let us imagine that the functions \(F_i\) embody a driving force which causes that variables \(x_i\) to move in one direction at \(t = t_0\) and in another direction at \(t = t_1\). In this case \(\gamma\) and \(\gamma'\) would be different, and might even cross at some point. (This does not violate the uniqueness of the trajectories, because the evolution times \(t - t_0\) and \(t - t_1\) would be different.) However if the functions \(F_i\) do not have an explicit time dependence, then the curve \(\gamma\) spanned by the trajectory which goes through the point \(x_i = x_{i,0}\) will be independent of the time at which the trajectory goes through this point. In other words, the curves spanned by the trajectories will be time independent and also, quite importantly, will be non overlapping. (If two different curves \(\gamma\) and \(\gamma'\) had a point in common, we could take this point as the initial point for the evolution. The trajectories would then be identical and \(\gamma\) and \(\gamma'\) would have to be identical as well.)

In what follows we will restrict our considerations to the case where the functions \(F_i\) have no explicit time dependence. It is then immaterial to distinguish between the curve \(\gamma\) spanned by a trajectory and the trajectory itself (in principle the trajectory would consist of \(\gamma\) plus the specification of the time at which the variables \(x_i\) go through a definite point of \(\gamma\)) and we will refer to the curves spanned in phase space as “the trajectories.”
In the specific case of a two dimensional phase space, which we will consider in the rest of these notes, a consequence of the uniqueness of the trajectories is that the phase space is divided into non-overlapping regions, each trajectory occupying one of these regions. The word “region” must be intended in a wide sense. If a trajectory is closed, as in the case of the harmonic oscillator where, taking \( x(t) \) as the coordinate and \( y(t) \) as the momentum, the trajectories are ellipses, the “regions” consist of single closed curves. In other cases, however, the trajectories may consist of open curves with the system evolving from one point of instability to a point of attraction, or a trajectory may wind in between two limiting cycles, in which case the region occupied by the trajectory would be the domain between the two limiting cycles.

**Fixed-points and linearization. Conservation laws.**

As mentioned above, with nonlinear ODEs what can be done with analytic techniques is, in general, quite limited, and one needs to resort to computational methods to study the properties of the solutions. There are a few things, however, which can be done by analytic methods. Let us consider first the possible fixed-points of the equations. Given the equations

\[
\frac{dx_i}{dt} = F_i(x_j(t)) \quad i = 1, 2, \ldots n \quad (12)
\]

(note that we assume no explicit time dependence) there may be one or more points \( x_j^{(0)} \) for which the functions \( F_i(x_j) \) are all equal to zero:

\[
F_i(x_j^{(0)}) = 0 \quad (13)
\]

Then, of course,

\[
x_i(t) = x_i^{(0)} = \text{constant} \quad (14)
\]

solves the equations. The points \( x_j^{(0)} \) with this property are called “fixed-points” of the equations. One can then expand the equations to first order about the fixed-points. Let us set

\[
x_i(t) = x_i^{(0)} + y_i(t) \quad (15)
\]

and expand the equations for small \( y_i(t) \). We get

\[
\frac{dy_i}{dt} = F_i(x_j^{(0)} + y_j(t)) = \sum_j \frac{\partial F_i(x_j)}{\partial x_j} \bigg|_{x_j^{(0)}} y_j(t) + O(y^2) \quad (16)
\]

Let us denote by \( A_{i,j} \) the matrix of derivatives in Eq. 16

\[
A_{i,j} = \frac{\partial F_i(x_j)}{\partial x_j} \bigg|_{x_j^{(0)}} \quad (17)
\]

Then, neglecting terms of higher order in \( y_i \), the evolution equations for the variables \( y_i(t) \) become the set of linear equations with fixed coefficients

\[
\frac{dy_i}{dt} = \sum_j A_{i,j} y_j(t) \quad (18)
\]
which is amenable to an analytic solution. This solution might then show that, as $t$ increases, the variables $y_i(t)$ tend to zero, in which case the fixed-point is a “stable fixed point”: if the system is perturbed away from the fixed-point it will return to it; or that the variables $y_i(t)$ tend to infinity, in which case the fixed-point is an “unstable fixed point”: a small perturbation from the fixed-point will drive the system away from it. There are other possibilities, for example the solution to Eqs. 18 may be of cyclic nature, with the system circling around the fixed-point without moving toward it or away from it. The behavior of the solutions in the neighborhood of a fixed-point will depend on the system under consideration, but the point we are making here is that it can be investigated by analytic means.

Another situation where the properties of the system can be studied to some extent analytically occurs if the non-linear terms in the equations are proportional to a parameter $\epsilon$ in such a way that for $\epsilon = 0$ the equations become linear. In this case some analytical insight into the properties of the system may be obtained through an expansion of the equations in powers of $\epsilon$.

Finally let us mention that in some cases it is possible to establish analytically some conservation laws, like the conservation of energy in time independent Hamiltonian systems, which can also provide insights into the properties of the solutions.

This brief discussion does not, by any means, exhaust what can be done with analytical methods in the study of non-linear ODEs. Indeed, while today we have the possibility enlisting the aid of powerful computers in their investigations, many interesting results have been obtained on a variety of non-linear equations before they could be studied by computational techniques.

In the rest of these lecture notes we will consider in some detail two nonlinear ODEs, as an illustration of the interesting properties of the solutions to non-linear ODEs.

**The “Lotka-Volterra equation”**.

The “Lotka-Volterra equation” is a nonlinear ODE in two dimensional phase space which can be used to describe the behavior of a predator-prey system\(^1\). The equations are

\[
\begin{align*}
\frac{dx}{dt} &= ax - bxy \\
\frac{dy}{dt} &= dxy - cy
\end{align*}
\]  

(19) (20)

where $x$ represents the number of prey, $y$ the number of predators, and $a, b, c, d$ are positive parameters\(^2\). The rationale behind the equations is that in absence of interaction between

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\(^{1}\)Proposed in 1926 by Vito Volterra to model the changes in fish populations in the Adriatic sea and, independently, in 1925 by Alfred Lotka to model a hypothetical chemical reaction where the components oscillate (from http://www.scholarpedia.org/).

\(^{2}\)Although Eqs. 19 and 20 appear to depend on four parameters, by rescaling the variables $x$ and $y$ and the time coordinate $t$ one can set three of the parameters equal to 1, so effectively the system of equations only depends on one parameter.

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the species (i.e. with $b = d = 0$) the prey would grow exponentially ($dx/dt = ax$ is solved by $x(t) = x(0)e^{at}$) and the predators would die ($dy/dt = -cy$ is solved by $y(t) = y(0)e^{-ct}$.) However, the rate of decay of the number of predators is reversed, leading to their growth, if $x$ is sufficiently large to make $dx - c > 0$. With these equations it turns out that the number of prey will always continue to grow until the number of predators begins to grow. At some point the growth of $y$ will lead to a reversal of the rate of growth $a - by$ of the prey, which will start to decrease in number to the point when this will stymie the growth of the predator population which will begin to decrease as well, and the whole affair will result in a cyclic behavior.

![Figure 1: Solutions to the Lotka-Volterra equation with $a, b, c, d$ all set equal to 1. The figure at left shows the time dependence of the solution with initial values $x(0) = y(0) = 0.5$: the initial value of $x$ (the prey) is not sufficient to stop the decrease of $y$ (the predators), so the number of predators decrease while the prey population grows, but eventually the increase in the prey becomes sufficient to reverse the decrease in the number of predators, which begins to grow. The figure at right shows the trajectories in phase space, with $x$ and $y$ in the horizontal and vertical axes, for initial values $x(0) = y(0) = 0.5, 0.7, 0.9$. The fixed-point at $x = y = 1$ is also shown in the figure. In the neighborhood of the fixed-point the trajectories approach those of harmonic motion.](image)

The equations have two fixed-points, obtained by demanding $dx/dt = 0$ and $dy/dt = 0$. This leads to the equations

\[
\begin{align*}
ax - bxy &= 0 \\
axy - cy &= 0
\end{align*}
\]

which are solved by $x_1 = y_1 = 0$ and $x_2 = c/d, y_2 = a/b$. We can investigate the stability of the two fixed-points by linearizing the equations around them. For the first solution, the
linearized equations are obviously

\[
\frac{dx}{dt} = ax \tag{23}
\]

\[
\frac{dy}{dt} = -cy \tag{24}
\]

which is unstable in the direction of growing \( x \). So the equations tell us that, with zero predators, a minimal number of prey will lead away from the fixed point.

In order to linearize the second solution let us define new variables \( \delta x = x - c/d, \delta y = y - a/b \). Substituting into the equations these take the form

\[
\frac{d\delta x}{dt} = a\left(\delta x + \frac{c}{d}\right) - b\left(\delta x + \frac{c}{d}\right)\left(\delta y + \frac{a}{b}\right) \tag{25}
\]

\[
\frac{d\delta y}{dt} = d\left(\delta x + \frac{c}{d}\right)\left(\delta y + \frac{a}{b}\right) - c\left(\delta y + \frac{a}{b}\right) \tag{26}
\]

or, keeping only terms of the first order in \( \delta x, \delta y \),

\[
\frac{d\delta x}{dt} = -\frac{bc}{d} \delta y \tag{27}
\]

\[
\frac{d\delta y}{dt} = \frac{ad}{b} \delta x \tag{28}
\]

These are the equations of a harmonic oscillators with angular frequency \( \sqrt{ac} \), which shows that the second fixed point is stable. A small displacement from the fixed point will lead to oscillations around it.

While for bigger displacements from the fixed point an analytic solution to the equations of motion cannot be given, it is possible however to find the shape the trajectory. Indeed by dividing Eq. 20 by Eq. 19 we find

\[
\frac{dy}{dx} = \frac{dxy - cy}{ax - bxy} \tag{29}
\]

or

\[
(ax - bxy)\, dy = (dxy - cy)\, dx \tag{30}
\]

By dividing by \( xy \) this gives

\[
\frac{a - by}{y} \, dy = \frac{dx - c}{x} \, dx \tag{31}
\]

and

\[
\left(\frac{a}{y} - b\right) \, dy = \left(d - \frac{c}{x}\right) \, dx \tag{32}
\]

which can be integrated to give

\[
a \log y - by = dx - c \log x + K \tag{33}
\]
where $K$ is a constant depending on the initial conditions.

The Lotka-Volterra equation can be generalized to more than two species and to include a variety of interactions among the the species. With quadratic interaction terms, the most general form of the equations would be

$$\frac{dx_i(t)}{dt} = a_i x_i(t) + \sum_j A_{ij} x_i(t)x_j(t) \quad (34)$$

where the vector $\vec{a}$ describes the behavior (growth or decline) of the species without mutual interactions, while the matrix $A$ describes the interactions among species. The solutions can exhibit a wide variety of behaviors, including fixed-points, limiting cycles and chaos, determined by the parameters $a_i$ and $A_{ij}$.

The “Van der Pol” equation.

The equation of Van der Pol

$$\frac{d^2 x}{dt^2} - \epsilon(1 - x^2) \frac{dx}{dt} + ax = 0, \quad \text{with } a > 0, \ \epsilon \geq 0 \quad (35)$$

is a second order, nonlinear differential equation whose solutions exhibit interesting oscillations and a limit cycle\(^3\). With the rescaling $t \rightarrow t/\sqrt{a}$, $\epsilon \rightarrow \epsilon/\sqrt{a}$ the parameter $a$ may be eliminated from the equation and we will proceed with $a = 1$. By introducing $y(t) = dx(t)/dt$ as a new variable, the equation can be recast as the following system of two first order, nonlinear equations

$$\frac{dx}{dt} = y \quad (36)$$
$$\frac{dy}{dt} = \epsilon(1 - x^2)y - x \quad (37)$$

The system has a fixed-point for $x, y$ satisfying the equations

$$\begin{align*}
y &= 0 \quad (38) \\
\epsilon(1 - x^2)y - x &= 0 \quad (39)
\end{align*}$$

with the obvious solution

$$x = y = 0 \quad (40)$$

Linearizing the equations around the fixed-point, i.e. the origin, we find the equations

$$\begin{align*}
\frac{dx}{dt} &= y \quad (41) \\
\frac{dy}{dt} &= \epsilon y - x \quad (42)
\end{align*}$$

\(^3\)The equation was published in 1926 by Balthasar van der Pol when he was working at Philips on vacuum tubes and found stable current oscillations (from https://en.wikipedia.org/).
or, in matrix form

\[
\begin{pmatrix}
\frac{dx}{dt} \\
\frac{dy}{dt}
\end{pmatrix} = M \begin{pmatrix} x \\ y \end{pmatrix}
\]  \hspace{1cm} (43)

with

\[
M = \begin{pmatrix} 0 & 1 \\ -1 & \epsilon \end{pmatrix}
\]  \hspace{1cm} (44)

Solving the equation

\[
\det(M - \lambda I) = \lambda(\lambda - \epsilon) + 1 = \lambda^2 - \epsilon\lambda + 1 = 0
\]  \hspace{1cm} (45)

we find the eigenvalues

\[
\lambda_{\pm} = \frac{\epsilon \pm \sqrt{\epsilon^2 - 4}}{2}
\]  \hspace{1cm} (46)

For \( \epsilon < 2 \) we have

\[
\lambda_{\pm} = \frac{\epsilon \pm i\sqrt{4 - \epsilon}}{2}
\]  \hspace{1cm} (47)

which shows that the vector \((x(t), y(t))\) will have a magnitude growing as \(\exp(\epsilon t/2)\) and an oscillatory component with angular velocity \(\omega = \sqrt{1 - (\epsilon/2)^2}\). For \( \epsilon \geq 2 \) the two eigenvalues will be positive and, again, the magnitude of \((x(t), y(t))\) will grow exponentially. So in either case the fixed-point is an unstable fixed point.

With \( \epsilon = 0 \) the Van der Pol equations 36, 37 become the linear equations

\[
\begin{align*}
\frac{dx}{dt} &= y \\
\frac{dy}{dt} &= -x
\end{align*}
\]  \hspace{1cm} (48, 49)

which describe the motion of a harmonic oscillator with \(\omega = 1\). The trajectories spanned by the solutions are circles, which can have any radius \(R\). When one moves away from the limit \(\epsilon = 0\) one must resort to computational methods to investigate the solutions. It is however possible to start from the harmonic oscillator solutions with \(\epsilon = 0\) and expand the equations to first order in \(\epsilon\). The resulting equations can be solved analytically. To derive them let us start from the solution

\[
\begin{align*}
x(t) &= R \cos t \\
y(t) &= -R \sin t
\end{align*}
\]  \hspace{1cm} (50, 51)

and look for a solution of the form

\[
\begin{align*}
x(t) &= R \cos t + \epsilon \tilde{x}(t) \\
y(t) &= -R \sin t + \epsilon \tilde{y}(t)
\end{align*}
\]  \hspace{1cm} (52, 53)
Figure 2: At left: computer generated phase space trajectories for $\epsilon = 0.01$: with initial data $x(0) = 0, y(0) = 2$ the trajectory, shown in blue, follows a limit cycle of radius $R \approx 2$; with initial data $x(0) = 0, y(0) = 3$ the trajectory, shown in red, spirals inwards toward the limit cycle; with initial data $x(0) = 0, y(0) = 1$ the trajectory, shown in green, spirals outwards toward the limit cycle. At right: limit cycles for $\epsilon = 0.5$, shown in blue, $\epsilon = 1.5$, shown in green, and $\epsilon = 2.5$, shown in red.

Inserting these expressions into Eqs. 36, 37 we obtain the equations

$$\frac{d[R \cos t + \epsilon \tilde{x}(t)]}{dt} = -R \sin t + \epsilon \tilde{y}(t)$$

$$\frac{d[-R \sin t + \epsilon \tilde{y}(t)]}{dt} = \epsilon [1 - [R \cos t + \epsilon \tilde{x}(t)]^2] [-R \sin t + \epsilon \tilde{y}(t)] - R \cos t - \epsilon \tilde{x}(t)$$

Expanding to first order in $\epsilon$ we get

$$\frac{dR \cos t}{dt} + \epsilon \frac{d\tilde{x}(t)}{dt} = -R \sin t + \epsilon \tilde{y}(t)$$

$$-\frac{dR \sin t}{dt} + \epsilon \frac{d\tilde{y}(t)}{dt} = \epsilon [-R \sin t + R^3 \cos^2 t \sin t] - R \cos t - \epsilon \tilde{x}(t)$$

The term of order zero cancel, of course, and we are left with

$$\frac{d\tilde{x}(t)}{dt} = \tilde{y}(t)$$

$$\frac{d\tilde{y}(t)}{dt} = -\tilde{x}(t) - R \sin t + R^3 \cos^2 t \sin t$$
At this point it is convenient to go back to a single ODE of the second order. Differentiating Eq. 58 we get
\[ \frac{d\tilde{y}(t)}{dt} = \frac{d^2\tilde{x}(t)}{dt^2} \] (60)
and inserting this into Eq. 59 we find
\[ \frac{d^2\tilde{x}(t)}{dt^2} + \tilde{x}(t) = -R \sin t + R^3 \cos^2 t \sin t = -R \sin t + R^3 \frac{\sin 3t + \sin t}{4} \] (61)
or
\[ \frac{d^2\tilde{x}(t)}{dt^2} + \tilde{x}(t) = -\left[ R - \frac{R^3}{4} \right] \sin t + \frac{R^3}{4} \sin 3t \] (62)
This is a second order, non-homogeneous differential equation which can easily be solved. The solution, which is left as an exercise, shows that if \( R = 2 \) the perturbation of the zero order solution is stable, while if \( R \) is larger or smaller than 2 the magnitude of the perturbation increases with time. Moreover if \( R > 2 \) the perturbation moves the trajectory toward smaller \( R \), while it moves it toward larger \( R \) if \( R < 2 \). Thus something remarkable happens: with \( \epsilon = 0 \) the trajectories are circles and are stable for any \( R \), but as soon as the parameter \( \epsilon \) becomes larger than zero, however small, the non-linearity pushes the trajectory toward a limit cycle, which for very small \( \epsilon \) is very close to a circle of radius 2. This is illustrated in the left image in Fig. 2 which shows the phase space evolution of three trajectories with \( \epsilon = 0.01 \).

Figure 3: Phase space trajectory (left image) and time evolution (right image) for a computer generated solution to the Van der Pol equation with \( \epsilon = 1 \). In the right image \( x(t) \) is shown in red and \( y(t) \) in green.

When one moves away from the linear limit, the solutions of the Van der Pol equation must be obtained with computational methods. They exhibit a limit cycle for all values of \( \epsilon > 0 \). The image at right in Fig. 2 shows three limit cycles.
Figure 3 shows the phase space trajectory and the evolution in time of a solution with $\epsilon = 1$ and initial data $x(0) = 0, y(0) = 0.1$. The onset of a cyclic behavior is apparent.

In all the figures, and especially in the right image of Fig. 2, one may notice that the excursion of the $x$ variable in the limit cycles is between $-2$ and $2$, irrespective of the value of $\epsilon$. The maximum of $|x(t)|$ is not exactly 2, but it is equal to 2 to a very good approximation. I do not have, nor could I find, any explanation for this curious feature of the limit cycles.