Complex numbers:
\[ z = x + iy = r(\cos \phi + i \sin \phi) \] (1)

\( x = \text{Re } z \): the real part of \( z \)
\( y = \text{Im } z \): the imaginary part of \( z \)
\( r \): the modulus, or magnitude, or absolute value of \( z \)
\( \phi \): the argument or phase of \( z \)

Complex conjugate
\[ z^* = x - iy = r(\cos \phi - i \sin \phi) \] (2)

Arithmetic of complex numbers (mostly trivial)
\[ zz^* = r^2 \] (3)
\[ z^{-1} = \frac{1}{z} = \frac{z^*}{zz^*} = \frac{x - iy}{r^2} \] (4)

Note: the complex numbers \( \mathbb{C} \) are a field extension of the real numbers \( \mathbb{R} \): \( \mathbb{C} = \mathbb{R}(i) \).

Euler’s formula:
\[ e^{i\phi} = \cos \phi + i \sin \phi \] (5)

From Euler’s formula
\[ e^z = e^{x+iy} = e^x(\cos y + i \sin y) \] (6)

Exercises:
- use Euler’s formula to calculate \( \sin(\alpha + \beta) \), \( \cos(\alpha + \beta) \);
- use Euler’s formula to prove De Moivre’s formula \((\cos \phi + i \sin \phi)^n = \cos n\phi + i \sin n\phi\);
- use De Moivre’s formula to express \( \sin 3\phi \) and \( \cos 3\phi \) in terms of powers of \( \sin \phi \) and \( \cos \phi \); show that \( \cos n\phi \) can be expressed in terms of powers of \( \cos \phi \), and that \( \sin n\phi \) can be expressed as \( \sin \phi \) times a polynomial of degree \( n - 1 \) in \( \cos \phi \).

Functions in the complex plane, analytic functions:

For a general function
\[ f = f(x, y) = f(z, z^*) \] (7)

For a general function one must allow a dependence on \( z^* \) as well as \( z \). The requirement that \( f \) depends only of \( z \), i.e. \( f = f(x + iy) \) is very restrictive, yet extremely important, see the rest of these notes.
Examples.

There are three independent homogeneous polynomials of degree 2, e.g.

\[ x^2, \quad xy, \quad y^2 \]  

(8)

or, otherwise,

\[ z^2 = x^2 - y^2 + 2ixy, \quad zz^* = x^2 + y^2, \quad (z^*)^2 = x^2 - y^2 - 2ixy \]  

(9)

None of the three polynomials in Eq. 8 can be expressed in terms \( z \) alone. The most general homogeneous polynomial of degree 2 expressible in terms of \( z \) alone is

\[ p_{2, h}(z) = cz^2 \]  

(10)

where \( c \) is an arbitrary complex constant.

In general, a non-homogeneous polynomial, of degree \( n \), function of \( z \) only will be given by

\[ p_n(z) = \sum_{k=0}^{n} c_k z^k \]  

(11)

If we substitute \( x + iy \) for \( z \) we will obtain a polynomial of degree \( n \) in \( x \) and \( y \) with complex coefficients of a form much less general than an arbitrary polynomial of degree \( n \) in \( x \) and \( y \) (with complex coefficients.) Polynomials which are only functions of \( z \), as in Eq. 11, are “analytic” polynomials.

The above can be generalized to functions. Loosely speaking, a function \( f \) in the complex plane is analytic if it is only a function of \( z \) (and not a function of \( z \) and \( z^* \).)

The polynomial expansion of Eq. 11 can be generalized to a power series expansion

\[ f(z) = \sum_{k=0}^{\infty} c_k z^k \]  

(12)

If the power series converges it defines an analytic function over its domain of convergence. Functions with a domain of convergence that covers the whole complex plane are called “entire functions”. The exponential function \( \exp(z) \), for example, is an entire function. If the power series expansion of \( f(z) \) has a finite domain of convergence and if the point \( z = a \) is within its domain of convergence, \( f(z) \) can be expanded about \( a \)

\[ f(z) = \sum_{k=0}^{\infty} \tilde{c}_k (z - a)^k \]  

(13)

where the expansion coefficients will be different. Equation 13 defines the Taylor series expansion of the analytic function \( f(z) \) about \( z = a \). It is likely that the series 13 will have a domain of convergence extending beyond the one of the series 12 and thus it will define
an analytic continuation of that series. Continuing in this fashion, by progressive analytic continuations, one can extend the analytic function throughout the complex plane, apart from points of singularity where it is not defined. We will follow, however, a different route to the definition of an analytic function.

The Cauchy-Riemann conditions:

Consider the function

\[ f(x, y) = u(x, y) + iv(x, y) \]  

(14)

We have

\[ df(x, y) = \frac{\partial u(x, y)}{\partial x} dx + \frac{\partial u(x, y)}{\partial y} dy + i\frac{\partial v(x, y)}{\partial x} dx + i\frac{\partial v(x, y)}{\partial y} dy = \]

\[ \left( \frac{\partial u(x, y)}{\partial x} + i\frac{\partial v(x, y)}{\partial x} \right) dx + \left( \frac{\partial v(x, y)}{\partial y} - i\frac{\partial u(x, y)}{\partial y} \right) (i dy) \]  

(15)

On the other hand, if \( f \) is a function of \( z \) only we will have

\[ df(z) = f'(z) dz = f'(z) (dx + i dy) \]  

(16)

where the complex number \( f'(z) \) can be identified with the derivative of \( f(z) \) with respect to \( z \). Comparing Eqs. 15 and 16 we see that a necessary and sufficient condition for \( f(x, y) \) to be a function of \( z \) only is that in a neighborhood of \( z \)

\[ \frac{\partial u(x, y)}{\partial x} + i\frac{\partial v(x, y)}{\partial x} = \frac{\partial v(x, y)}{\partial y} - i\frac{\partial u(x, y)}{\partial y} \]  

(17)

Then the common value of the two sides of the equation will give the derivative of \( f \) at \( z \)

\[ \frac{df(z)}{dz} = \frac{\partial u(x, y)}{\partial x} + i\frac{\partial v(x, y)}{\partial x} = \frac{\partial v(x, y)}{\partial y} - i\frac{\partial u(x, y)}{\partial y} \]  

(18)

Equating the real and imaginary parts in the two sides of Eq. 17 we obtain the Cauchy-Riemann conditions:

\[ \frac{\partial u(x, y)}{\partial x} = \frac{\partial v(x, y)}{\partial y} \]  

(19)

\[ \frac{\partial v(x, y)}{\partial x} = -\frac{\partial u(x, y)}{\partial y} \]  

(20)

When these are satisfied throughout a certain domain \( D \) of the complex plane the function \( f(x, y) = u(x, y) + iv(x, y) \) is an analytic function \( f(z) \) and, viceversa, if \( f \) is an analytic function \( f(z) \) in \( D \) then the real and imaginary parts of \( f \) satisfy the Cauchy-Riemann conditions.
Properties of analytic functions: Cauchy’s theorem.

Given a closed curve $\gamma$ within a simply connected domain $D$ where $f(z) = u(x, y) + iv(x, y)$ is analytic, consider the integral

$$ I = \oint_{\gamma} f(z) \, dz $$

(21)

Mathematically this is defined by specifying $\gamma$ through the radius vector $\vec{r} = (x, y)$ pointing to a point of $\gamma$ and letting $\vec{r} = \vec{r}(s)$ depend on a parameter $s$ which runs between $s_0$ and $s_1$ as $\vec{r}(s)$ describes the whole curve:

$$ I = \int_{s_0}^{s_1} [u(x(s), y(s)) + v(x(s), y(s))](\frac{dx(s)}{ds} + i \frac{dy(s)}{ds}) \, ds = I_{\text{Re}} + i I_{\text{Im}} $$

(22)

with

$$ I_{\text{Re}} = \int_{s_0}^{s_1} \left( u(x(s), y(s)) \frac{dx(s)}{ds} - v(x(s), y(s)) \frac{dy(s)}{ds} \right) \, ds $$

(23)

$$ I_{\text{Im}} = \int_{s_0}^{s_1} \left( v(x(s), y(s)) \frac{dx(s)}{ds} + u(x(s), y(s)) \frac{dy(s)}{ds} \right) \, ds $$

(24)

Note that if we define the vector fields

$$ \vec{w}_1(x, y) = (u(x, y), -v(x, y)) $$

(25)

$$ \vec{w}_2(x, y) = (v(x, y), u(x, y)) $$

(26)

then

$$ I_{\text{Re}} = \int_{\gamma} \vec{w}_1 \cdot d\vec{r} $$

(27)

$$ I_{\text{Im}} = \int_{\gamma} \vec{w}_2 \cdot d\vec{r} $$

(28)
By Stokes theorem

\[ I_{\text{Re}} = \int_{\Sigma} (\text{curl } \vec{w}_1) \, d\sigma \]  
\[ I_{\text{Im}} = \int_{\Sigma} (\text{curl } \vec{w}_2) \, d\sigma \]

(29)  
(30)

where \( \Sigma \) is the surface in the complex plane enclosed by \( \gamma \). Now we have (see “Vector operators in 2-d” in Lecture 1)

\[
\text{curl } \vec{w}_1 = \frac{\partial w_{1,y}}{\partial x} - \frac{\partial w_{1,x}}{\partial y} = -\frac{\partial v(x,y)}{\partial x} - \frac{\partial u(x,y)}{\partial y}
\]

(31)

\[
\text{curl } \vec{w}_2 = \frac{\partial w_{2,y}}{\partial x} - \frac{\partial w_{2,x}}{\partial y} = \frac{\partial u(x,y)}{\partial x} - \frac{\partial v(x,y)}{\partial y}
\]

(32)

but on account of the Cauchy-Riemann conditions both curl vanish. Therefore: “the contour integral of an analytic function along any curve contained in a simply connected domain where the function is analytic will vanish.” (Cauchy’s theorem)

Properties of analytic functions: Cauchy’s integral formula.

Consider a closed curve \( \gamma \) within a domain \( D \) where \( f(z) \) is analytic and where \( \gamma \) can be continuously deformed into a small circle of radius \( \epsilon \) with center in \( z \). The function

\[ g(w) = \frac{f(w)}{w - z} \]

(33)

is not analytic in \( D \) as a function of \( w \) because it has a singularity for \( w = z \), but is it analytic over a domain \( D' \) which excludes an arbitrarily small circle around \( z \). Consider then the
closed contour formed by $\gamma$ in counterclockwise direction, a path $p$ leading from any point of $\gamma$ to a point on a circle of small radius $\epsilon$ with center in $z$, the circle $c$ (or more properly its circumference) in clockwise direction, and the path $p$ traversed now in the opposite direction (see Fig. 2, at left.) By Cauchy’s theorem, applied to this path, we have

$$I = \oint_{\gamma} g(w) \, dw + \oint_{c} g(w) \, dw = 0$$ (34)

since $g(w)$ is analytic in a domain enclosing the path and the two contributions from the integrals along $p$ cancel. As a consequence

$$\oint_{\gamma} \frac{f(w)}{w - z} \, dw = - \oint_{c} \frac{f(w)}{w - z} \, dw = 0$$ (35)

Since $\epsilon$ can be taken arbitrarily small, $f(w)$ over $c$ may be replaced by $f(z)$ with an error which will go to zero as $\epsilon \to 0$. We deduce

$$\oint_{\gamma} \frac{f(w)}{w - z} \, dw = -f(z) \oint_{c} \frac{1}{w - z} \, dw$$ (36)

The integral in the r.h.s. of this equation can be calculated by setting

$$w = z + \epsilon e^{-i\phi}$$ (37)

which gives

$$- \oint_{c} \frac{1}{w - z} \, dw = - \int_{0}^{2\pi} \frac{1}{\epsilon e^{-i\phi}} \, d(z + \epsilon e^{-i\phi}) = 2\pi i$$ (38)

Finally one obtains “Cauchy’s integral formula”:

$$f(z) = \frac{1}{2\pi i} \oint_{\gamma} \frac{f(w)}{w - z} \, dw$$ (39)

where $\gamma$ is an arbitrary closed curve contained within the domain of analyticity of $f$ and circling $z$ only once.

**Corollary: deformation of a closed contour of integration.**

An argument very similar to the one used above to prove Cauchy’s integral formula can be used to prove that the contour of integration of an analytic function can be deformed without changing the value of the integral provided that the deformation occurs entirely within a domain of analyticity of the function. Consider the integrals

$$I = \oint_{\gamma} f(z) \, dz \quad \text{and} \quad I' = \oint_{\gamma'} f(z) \, dz$$ (40)

where $\gamma$ and $\gamma'$ are two closed contours which can be continuously deformed into each other remaining within a domain of analyticity of $f(z)$ (see Fig. 2, at right, where the inner contour
is labeled $-\gamma'$ because it is oriented in the opposite direction to the contour $\gamma'$.) We can join the two contours $\gamma$ and $-\gamma'$ with two infinitesimally close segments $p$ and $p'$ going in opposite directions, as shown in the figure, to form a single closed contour $\gamma \rightarrow p' \rightarrow \gamma' \rightarrow p$. The function $f(z)$ is analytic inside this contour and so we will have

$$\oint_{\gamma} f(z) \, dz + \oint_{p'} f(z) \, dz + \oint_{-\gamma'} f(z) \, dz + \oint_{p} f(z) \, dz = 0 \tag{41}$$

But the sum $\int_{p'} f(z) \, dz + \int_{p} f(z) \, dz$ vanishes because it consists of the integral of $f(z)$ along two identical segments traversed in opposite directions, and thus we are left with

$$\oint_{\gamma} f(z) \, dz + \oint_{-\gamma'} f(z) \, dz = \oint_{\gamma} f(z) \, dz - \oint_{\gamma'} f(z) \, dz = I - I' = 0 \tag{42}$$

which shows that the two contour integral $I$ and $I'$ are indeed identical.

Taylor series expansion and Laurent series expansion.

Let a circle $c$ of center $a$ and radius $r$ be within the domain of analyticity of $f(z)$. Then for $|z - a| < r$:

$$f(z) = \frac{1}{2\pi i} \int_{c} \frac{f(w)}{w - z} \, dw = \frac{1}{2\pi i} \int_{c} \frac{f(w)}{w - a - (z - a)} \, dw = \frac{1}{2\pi i} \int_{c} \frac{f(w)}{(w - a) \left(1 - (z - a)/(w - a)\right)} \, dw \tag{43}$$

Since $|(z - a)/(w - a)| < 1$ the fraction can be expanded analyticity of $f(z)$. Then for
\( |z - a| < r: \)

\[
f(z) = \frac{1}{2\pi i} \oint_{c} \frac{f(w)}{(w-a)} \left( 1 + \frac{z-a}{w-a} + \frac{(z-a)^2}{(w-a)^2} + \frac{(z-a)^3}{(w-a)^3} + \ldots \right) dw =
\]

\[
\frac{1}{2\pi i} \sum_{n=0}^{\infty} \left[ (z-a)^n \oint_{c} \frac{f(w)}{(w-a)^{n+1}} dw \right]
\]

(44)

or

\[
f(z) = c_0 + c_1(z-a) + c_2(z-a)^2 + c_3(z-a)^3 + \cdots = \sum_{n=0}^{\infty} c_n (z-a)^n
\]

(45)

with

\[
c_n = \frac{1}{2\pi i} \oint_{c} \frac{f(w)}{(w-a)^{n+1}} dw
\]

(46)

Equations 45 and 46 give the Taylor series expansion of \( f(z) \) around \( z = a \). By taking the \( n^{th} \) derivative of the series and then setting \( z = a \) one can see that

\[
c_n = \frac{1}{n!} \left. \frac{d^n f(z)}{dz^n} \right|_{z=a}
\]

(47)

as can also be seen by differentiating Cauchy’s integral formula Eq. 39 \( n \) times under the integral sign.

Consider now the annular region \( A \) between two circles \( c \) and \( C \) with center in \( a \) and radii \( r < R \). Let \( f(z) \) be a function which is analytic in a domain which encloses \( A \), but may have singularities inside the smaller circle. Apply Cauchy’s integral formula to the contour formed by the outer circle \( C \) in counterclockwise direction, a path \( p \) from any point of \( C \) to a point of \( c \), the inner circle \( c \) in clockwise direction, and the path \( p \) in opposite direction to get back to \( C \). For any point \( z \) in \( A \) we have

\[
f(z) = \frac{1}{2\pi i} \oint_{c} \frac{f(w)}{w-z} dw + \frac{1}{2\pi i} \oint_{C} \frac{f(w)}{w-z} dw
\]

(48)

Consider the two terms in the r.h.s. separately. For the first term we can follow the steps used to derive Eqs. 45 and 46 to get

\[
\frac{1}{2\pi i} \oint_{c} \frac{f(w)}{w-z} dw = \sum_{n=0}^{\infty} c_n (z-a)^n
\]

(49)

with

\[
c_n = \frac{1}{2\pi i} \oint_{c} \frac{f(w)}{(w-a)^{n+1}} dw
\]

(50)

For the second term we can write it as

\[
\frac{1}{2\pi i} \oint_{c} \frac{f(w)}{w-z} dw = -\frac{1}{2\pi i} \oint_{c} \frac{f(w)}{z-a-(w-a)} dw = \frac{1}{2\pi i} \oint_{c} \frac{f(w)}{z-a-(w-a)} dw = \frac{1}{2\pi i} \oint_{c} \frac{f(w)}{1-(w-a)/(z-a)} dw
\]

(51)
where \(c'\) is now the same circular path but in counterclockwise direction. On \(c'\) we have 
\[ \left| \frac{w-a}{z-a} \right| < 1 \]
and so we may expand the fraction in the last term to get
\[
\frac{1}{2\pi i} \oint_{c'} \frac{f(w)}{w-z} dw = \frac{1}{2\pi i} \oint_{c'} f(w) \left( 1 + \frac{w-a}{z-a} + \frac{(w-a)^2}{(z-a)^2} + \frac{(w-a)^3}{(z-a)^3} + \ldots \right) \, dw \tag{52}
\]
or
\[
\frac{1}{2\pi i} \oint_{c'} \frac{f(w)}{w-z} \, dw = \sum_{n=1}^{\infty} c_{-n}(z-a)^{-n} \tag{53}
\]
with
\[
c_{-n} = \frac{1}{2\pi i} \oint_{c'} f(w) (w-a)^{n-1} \, dw \tag{54}
\]
Finally we note that the integrands in Eqs. 50 and 54 are analytic within \(A\). Thus the contours can be deformed and, in particular, brought to coincide with a single circle \(\gamma\) within \(A\). One thus gets the more compact formula
\[
f(z) = \sum_{n=-\infty}^{\infty} c_n(z-a)^n \tag{55}
\]
with
\[
c_n = \frac{1}{2\pi i} \oint_{\gamma} \frac{f(w)}{(w-a)^{n+1}} \, dw \tag{56}
\]
Equations 55 and 56 give the Laurent series expansion of \(f(z)\), convergent within \(A\). The sum of the zero and positive terms in Eq. 55 form the “regular part” of the series, the sum of the negative terms form the “principal part of the series”. Note that if the principal part does not vanish the coefficient \(c_n\) with positive \(n\) cannot be related to the derivatives of \(f\) at \(z = a\), where \(f\) may not even be defined.

**Isolated singularities, residue at a singularity:**

Let \(f(z)\) be analytic in an open domain around the point \(a\) which only excludes \(a\). Then \(f(z)\) will have a Laurent series expansion Eq. 55 about \(a\).

If the principal part of the series vanishes, then the Laurent series expansion reduces to the Taylor series expansion
\[
f(z) = \sum_{n=0}^{\infty} c_n(z-a)^n \tag{57}
\]
and \(f(z)\) will be analytic in \(a\). If \(c_0 \neq 0\) then \(f(a) = c_0\). If \(c_0 = 0\) \(f(z)\) has a zero in \(a\). The zero will be a simple zero if \(c_1 \neq 0\). If the coefficients \(c_1, c_2 \ldots c_{m-1}\) vanish but \(c_m \neq 0\), then the expansion will begin with the term \(c_m(z-a)^m\) and \(f(z)\) will have a zero of order \(m\) at \(a\).

If the principal part of the Laurent series does not vanishes, \(f(z)\) is singular for \(z = a\) (or has an isolated singularity at \(a\)) and there are two cases:
If the principal part only contains a finite number of terms, i.e.

\[ f(z) = \sum_{n=-m}^{\infty} c_n (z - a)^n \]  

(58)

the \( f(z) \) has a pole of order \( m \) for \( z = a \). It will behave as

\[ f(z) = \frac{1}{(z - a)^m} g(z) \]  

(59)

where

\[ g(z) = \sum_{n=0}^{\infty} c_{n-m} (z - a)^n \]  

(60)

is analytic at \( z = a \).

If the principal part contains an infinite number of terms then the singularity of \( f(z) \) at \( a \) is an “essential singularity.” An example is the function \( f(z) = \exp[1/(z - a)] \) which has the Laurent series expansion

\[ f(z) = e^{1/(z-a)} = \sum_{n=0}^{\infty} \frac{1}{n!} (z - a)^{-n} \]  

(61)

and an essential singularity for \( z = a \).

**Residue at a singularity, theorem of residues:**

If the function \( f(z) \) has an isolated singularity at \( z = a \) the quantity

\[ \text{Res}(f, a) = \frac{1}{2\pi i} \oint_{\gamma} f(z) \, dz \]  

(62)

where \( \gamma \) is a small counterclockwise loop enclosing the singularity is the “residue” of \( f(z) \) at \( a \). The residue is equal to the coefficient \( c_{-1} \) in the Laurent series expansion of \( f(z) \) about \( a \).

If \( f(z) \) is analytic in a simply connected domain \( D \) apart from isolated singularities at \( a_i \), \( i = 1 \ldots m \) and \( \Gamma \) is a counterclockwise loop in \( D \) enclosing all the singularities, then, by deforming \( \Gamma \) to reduce it to small loops around all the singularities plus lines joining these loops one obtains the “theorem of residues”

\[ \oint_{\Gamma} f(z) \, dz = 2\pi i \sum_{i=1}^{m} \text{Res}(f, a_i) \]  

(63)

One can take advantage of this result, together with a judicious choice of \( \Gamma \), to calculate several definite integrals.
Figure 4: Illustration for the theorem of residues. The loop $\Gamma$ can be shrunk into the loop formed by: a small circle around $a_1$, the segment joining this circle to a small circle around $a_2$, the small circle around $a_2$; the segment, starting from the end point of the previous segment, joining this circle to a small circle around $a_2$; the small circle around $a_3$; and finally a segment, starting from the end point of the previous segment, and ending at the beginning of the first segment. The loop integral of $f(z)$ around $\Gamma$ is equal to the loop integral around this loop; however the integral of $f(z)$ along the three segments, shown in green, vanishes, because they form a loop enclosing a region where $f(z)$ is analytic. Thus the original loop integral of $f(z)$ around $\Gamma$ is equal to the sum of the loop integrals of $f(z)$ along the three small circles, shown in blue, around the isolated singularities of $f(z)$ at $a_1, a_2, a_3$.

Applications of the theorem of residues:

Calculate

$$I_1 = \int_{-\infty}^{\infty} \frac{1}{z^2 + a^2} \, dz$$  \hspace{1cm} (64)$$

Close the integral from $-R$ to $R$, with very large $R$ which will be sent to $\infty$, with a semicircle of radius $R$ in the upper half plane.

$$f(z) = \frac{1}{z^2 + a^2} = \frac{1}{2ia} \frac{1}{z - ia} - \frac{1}{2ia} \frac{1}{z + ia}$$ \hspace{1cm} (65)$$

has a simple pole at $z = ia$ within this contour with residue $1/(2ia)$. The theorem of residues gives then

$$I_1 = 2\pi i \frac{1}{2ia} = \frac{\pi}{a}$$ \hspace{1cm} (66)$$

(Closing the contour to $-i\infty$ picks up the residue at $z = -ia$ with the same result.)

$$I_2 = \int_{-\infty}^{\infty} \frac{\cos z}{z^2 + a^2} \, dz$$  \hspace{1cm} (67)$$
Rewrite $I_2$ as

$$I_2 = \text{Re} \left[ \int_{-\infty}^{\infty} \frac{e^{iz}}{z^2 + a^2} \, dz \right]$$  \hspace{1cm} (68)$$

and close the contour to $+i\infty$, which is legitimate and does not change the value of the integral since $\exp(iz) \to 0$ for $z \to i\infty$. The closed contour picks up the residue at $z = ia$

$$\text{Res} \left( \frac{e^{iz}}{z^2 + a^2} , ia \right) = \text{Res} \left( \frac{e^{iz}}{(z - ia)(z + ia)} , ia \right) = \left( \frac{e^{iz}}{z + ia} \right)_{z=ia} = \frac{e^{-a}}{2ia}$$  \hspace{1cm} (69)$$

and thus

$$I_2 = \text{Re} \left[ 2\pi i \frac{e^{-a}}{2ia} \right] = \pi e^{-a}$$  \hspace{1cm} (70)$$

Figure 5: Contours for integrals 1 and 2 (left) and 3 (right.)

$$I_3 = \int_{-\infty}^{\infty} \frac{\sin z}{z} \, dz$$  \hspace{1cm} (71)$$

$f(z) = \sin z/z$ is analytic at $z = 0$ (can be continued to $z = 0$ with value 1.) Deform the contour of integration from the real axis by making it take a very small detour below the real axis near $z = 0$. Denote this deformed contour by $C$. The integral does not change and thus

$$I_3 = \int_C \frac{\sin z}{z} \, dz = \int_C \frac{e^{iz} - e^{-iz}}{2iz} \, dz$$  \hspace{1cm} (72)$$

Since the contour avoids the point $z = 0$ we may write

$$I_3 = I_3^+ + I_3^-$$  \hspace{1cm} (73)$$

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with
\[ I_3^+ = \int_C \frac{\pm e^{\pm iz}}{2iz} \, dz \] (74)

Now we can close the contour to \(+ i\infty\) for \(I_3^+\) and to \(- i\infty\) for \(I_3^-\). \(\exp(-iz)/(2iz)\) is analytic on and inside the contour closed to \(- i\infty\) since the detour below the real axis avoids the pole at \(z = 0\): thus \(I_3^- = 0\). About \(I_3^+\) the contour closed to \(+ i\infty\) does enclose the pole at \(z = 0\). So from the theorem of residues we obtain
\[ I_3^+ = 2\pi i \text{Res} \left( \frac{e^{iz}}{(2iz)} , 0 \right) = 2\pi i \frac{1}{2i} = \pi \] (75)
and
\[ I_3 = \int_{-\infty}^{\infty} \frac{\sin z}{z} \, dz = I_3^+ + I_3^- = \pi \] (76)

**Multi-valued functions. Riemann sheets and branch cuts:**

The analytic continuation of an analytic function can give origin to multiply valued functions. A case in point is the function
\[ f(z) = \sqrt{z} \] (77)

Let us write
\[ z = r e^{i\phi} \] (78)

and agree that \(\sqrt{r}\) denotes the positive value of the square root of \(r\). Let us now continue \(z\) from an initial real value \(z = r\) through a circle of radius \(r\) by letting the phase \(\phi\) in Eq. 78 increase from 0 to 2\(\pi\). When \(\phi\) reaches 2\(\pi\) \(z\) returns to its original real value \(r\). Consider the corresponding continuation of the function \(f(z) = \sqrt{z}\). With \(z\) real and equal to \(r\), \(\sqrt{z}\) can be either \(\sqrt{r}\) (which we defined to be positive) or \(-\sqrt{r}\). Let us take the positive value and continue \(z\) from the original real value through the circle of radius \(r\). The corresponding continuation of \(f(z)\) will be
\[ f(z) = \sqrt{r} e^{i\phi/2} \] (79)

and we see that when \(z\) returns to the real axis with \(\phi \to 2\pi\) \(f(z)\) does not return to its original value \(\sqrt{r}\) but rather goes to
\[ f(z) = \sqrt{r} e^{i(2\pi)/2} = -\sqrt{r} \] (80)

If \(z\) is further continued along the circle of radius \(r\), \(f(z)\) will now take values
\[ f(z) = \sqrt{r} e^{i(\phi+2\pi)/2} = -\sqrt{r} e^{i\phi/2} \] (81)

\(f(z)\) will return to its original value \(\sqrt{r}\) only after \(z\) completes another full tour around the circle. These considerations show \(f(z) = \sqrt{z}\) is multiply-valued, taking two possible values for each \(z\) different from zero. A proper representation for \(f(z)\) will have it span two copies of the complex plane, taking one of the two values
\[ f(z) = \pm \sqrt{r} e^{i\phi/2}, \quad \text{with} \ 0 \leq \phi < 2\pi \] (82)
Figure 6: Analytic functions representation. Left: the function \( f(z) = \sqrt{1-z} \). The figure shows the first Riemann sheet with a cut along the real axis for \( z > 1 \). Right: the function \( f(z) = \log z \). The figure shows the first Riemann sheet with a cut along the negative real axis. \( \log z \) has a zero for \( z = 1 \) and is positive on the real axis for \( z > 1 \), negative between 1 and 0. In these images the phase of \( f(z) \) is color coded, with red for real positive followed by the colors of the rainbow as the phase increases from 0 to \( 2\pi \equiv 0 \). The intensity of \( f(z) \) shows its magnitude, decreasing to zero when the magnitude drops below a cut-off and goes to zero.

on each of them. These two copies of the complex plane are called “Riemann sheets.” Upon analytic continuation \( f(z) \) goes from one sheet to the other.

An analytic function may have an infinite multiplicity of values. A prototypical example is the function

\[
 f(z) = \log z
\]  

which is defined by the requirement

\[
 e^{\log z} = z
\]  

If

\[
 z = r e^{i\phi}, \quad \text{with } 0 \leq \phi < 2\pi
\]  

then any of the values

\[
 \log z = \log r + i(\phi + 2\pi n) \quad \text{with integer } n \text{ and real } \log r
\]  

will satisfy Eq. 84. We thus see that \( f(z) = \log z \) has an infinity of values and, correspondingly, an infinity of Riemann sheets. As \( z \) loops around the origin, \( \log z \) moves to the different sheets with \( n \) increasing (decreasing) by 1 when \( z \) completes a closed counterclockwise (clockwise) loop.
Frequently one separates the Riemann sheets by introducing a suitable cut in the complex plane\(^1\). This is illustrated in Fig. 6 which gives a color coded representation of the functions \(f(z) = \sqrt{1 - z}\) and \(f(z) = \log z\) in the cut complex plane.

Figure 7: The two Riemann sheets of the function \(f(z) = \sqrt{1 - z}\). The illustration shows the continuity of the function as one moves across the cut from one Riemann sheet to the other.

Different Riemann sheets are joined at the cut. One selects a principal branch of the function \(f(z)\) under consideration and uses the cut plane to represent it. If the analytic continuation of \(f(z)\) reaches the cut, then the continuation proceeds to another sheet. For example, \(f(z) = \sqrt{z}\) can be represented in the complex plane with a cut proceeding from \(0\) to \(-\infty\) along the negative real axis. We take as the principal branch of \(\sqrt{z}\) the one where \(\sqrt{z} = \sqrt{r}\) (positive, see above) for real, positive \(z = r\). If starting from the real axis we move on the upper half plane along the semicircle

\[
z = re^{i\phi} \quad \text{with } 0 \leq \phi \leq \pi \text{ and real } \log r
\]  

we will have

\[
f(z) = \sqrt{z} = \sqrt{re^{i\phi/2}}
\]  

and when we reach the cut along the negative real axis we will have

\[
\sqrt{-r} = i\sqrt{r}
\]  

On the other hand if we move on the lower half plane along the semicircle

\[
z = re^{-i\phi} \quad \text{with } 0 \leq \phi \leq \pi \text{ and real } \log r
\]  

\(^1\)There is some arbitrariness in the placing of the cuts, which one can take advantage of for the purpose of definite calculations.
we will have
\[ f(z) = \sqrt{z} = \sqrt{r} e^{-i\phi/2} \]  
(91)
and when we reach the cut along the negative real axis we will have
\[ \sqrt{-r} = -i\sqrt{r} \]  
(92)
Thus \( f(z) = \sqrt{z} \) exhibits a discontinuity across the cut. But the analytic function \( f(z) \) itself is not discontinuous. If we extend the paths beyond the cut \( f(z) \) will vary with continuity taking value
\[ f(z) = \sqrt{r} e^{i\phi/2} \quad \text{with } \pi \leq \phi \leq 2\pi \]  
(93)
in the continuation of the former path, and taking value
\[ f(z) = \sqrt{r} e^{-i\phi/2} \quad \text{with } \pi \leq \phi \leq 2\pi \]  
(94)
in the continuation of the latter path. There is no discontinuity in the analytic continuations, they just occur on different Riemann sheets. Note that when \( \phi = 2\pi \) both continuations reach the same value \( f(r) = -\sqrt{r} \), i.e. the negative value for the square root of \( z = r \).

Application:
Use the cut plane to calculate
\[ I = \int_0^{\infty} \frac{x^{\alpha-1}}{1 + x} \, dx \quad \text{with } 0 < \alpha < 1 \]  
(95)
We will calculate this integral using a suitable contour integral in the cut complex \( z \)-plane.
Consider the function
\[ f(z) = \frac{z^{\alpha-1}}{1 - z} \]  
(96)
We write \( z \) in terms of its magnitude \( r \geq 0 \) and phase \( \phi \):
\[ z = re^{i\phi} \]  
(97)
Substituting into Eq. 96 we get
\[ f(z) = \frac{r^{\alpha-1}e^{i(\alpha-1)\phi}}{1 - z} \]  
(98)
f(\( z \)) has a branch point singularity at \( z = 0 \) coming from the numerator \( z^{\alpha-1} \) (recall that \( 0 < \alpha < 1 \) so that \( \alpha - 1 \) is never an integer.) We make \( f(z) \) single-valued by cutting the complex \( z \)-plane along the negative real axis. This defines for us the first Riemann sheet. Consider now a closed counterclockwise contour \( \gamma \) going from \( -\infty \) to 0 on the upper side of the cut, going from 0 to \( -\infty \) on the lower side of the cut and returning to the original point through a circle at \( \infty \) as shown in Fig. 8, and the integral
\[ C = \oint_{\gamma} f(z) \, dz \]  
(99)
Figure 8: Contour for the integral in Eq. 99. The cut in the complex plane is shown in green.

taken along this path. Since the integral over the circle at $\infty$ vanishes, $C$ will be given by

$$C = C_+ + C_-$$  \hspace{1cm} (100)

where $C_+, C_-$ stand for the line integrals above and below the cut. Precisely, if denote by $f_+(z)$ and $f_-(z)$ the values taken by $f(z)$ on the upper and, respectively, lower sides of the cut, we will have

$$C_+ = \int_{-\infty}^{0} f_+(z) \, dz = -\int_{0}^{-\infty} f_+(z) \, dz$$  \hspace{1cm} (101)

$$C_- = \int_{0}^{-\infty} f_-(z) \, dz$$  \hspace{1cm} (102)

We must now find the values of $f_+(z)$ and $f_-(z)$ in the integrands. In our Riemann sheet $z$ is real and positive on the positive real axis, so its phase there is $\phi = 0$. If we continue $z$, for example along a semicircle in the top complex plane, until it reaches the upper side of the cut, its phase there will be $\phi = \pi$. If we continue $z$, instead, along a semicircle in the lower complex plane until it reaches the lower side of the cut, its phase there will be $\phi = -\pi$. Note that, insofar as $z$ is concerned, $\phi = \pi$ or $-\pi$ makes no difference, since with $z = re^{i\phi}$,
\[ z = r e^{\pm \pi} = -r, \] as it should be since we are along the negative \( z \)-axis. But it does make a difference for \( e^{i(\alpha - 1)\phi} \) in Eq. 98. Indeed with \( \phi = \pi \) we find

\[ f_+(z) = \frac{r^{\alpha - 1} e^{i(\alpha - 1)\pi}}{1 - z} \quad (103) \]

while with \( \phi = -\pi \) we get

\[ f_-(z) = \frac{r^{\alpha - 1} e^{-i(\alpha - 1)\pi}}{1 - z} \quad (104) \]

Substituting into Eqs. 101 and 102 we obtain

\[ C_+ = - \int_0^{-\infty} \frac{r^{\alpha - 1} e^{i(\alpha - 1)\pi}}{1 - z} \, dz = -e^{i(\alpha - 1)\pi} \int_0^{-\infty} \frac{r^{\alpha - 1}}{1 - z} \, dz \quad (105) \]

\[ C_- = \int_0^{-\infty} \frac{r^{\alpha - 1} e^{-i(\alpha - 1)\pi}}{1 - z} \, dz = e^{-i(\alpha - 1)\pi} \int_0^{-\infty} \frac{r^{\alpha - 1}}{1 - z} \, dz \quad (106) \]

where we moved the phase factors in front of the integrals since they are constant along the integration. To conclude we change variable of integration from \( z = -r \) to \( r \) which gives \( dz = -dr \), getting first

\[ C_+ = e^{i(\alpha - 1)\pi} \int_0^{\infty} \frac{r^{\alpha - 1}}{1 + r} \, dr \quad (107) \]

\[ C_- = -e^{-i(\alpha - 1)\pi} \int_0^{\infty} \frac{r^{\alpha - 1}}{1 + r} \, dr \quad (108) \]

and, finally,

\[ C = \oint f(z) \, dz = C_+ + C_- = \]

\[ [e^{i(\alpha - 1)\pi} - e^{-i(\alpha - 1)\pi}] \int_0^{\infty} \frac{r^{\alpha - 1}}{1 + r} \, dr = 2i \sin[(\alpha - 1)\pi] \int_0^{\infty} \frac{r^{\alpha - 1}}{1 + r} \, dr \quad (109) \]

But the integral over \( r \) is the same integral over \( x \) we want to calculate, see Eq. 95, so we have

\[ C = \oint f(z) \, dz = 2i \sin[(\alpha - 1)\pi] \int \quad (110) \]

On the other hand the function \( f(z) \) is analytic inside the domain enclosed by \( \gamma \) apart from the pole at \( z = 1 \) with residue \(-1\). So, from the theorem of residues, we get

\[ C = \oint f(z) \, dz = -2\pi i \quad (111) \]

Inserting this result into Eq. 110 we find

\[ -2\pi i = 2i \sin[(\alpha - 1)\pi] \int = -2i \sin(\alpha \pi) \int \quad (112) \]
Harmonic functions:
If $f(z = x + iy)$ is an analytic function in some simply connected domain $D$ its real part $u(x, y)$ and imaginary part $v(x, y)$ satisfy the Cauchy-Riemann conditions

\[
\frac{\partial u(x, y)}{\partial x} = \frac{\partial v(x, y)}{\partial y} \tag{114}
\]

\[
\frac{\partial v(x, y)}{\partial x} = -\frac{\partial u(x, y)}{\partial y} \tag{115}
\]

From these we get

\[
\frac{\partial^2 u(x, y)}{\partial x^2} + \frac{\partial^2 u(x, y)}{\partial y^2} = \frac{\partial}{\partial x} \frac{\partial v(x, y)}{\partial y} - \frac{\partial}{\partial y} \frac{\partial v(x, y)}{\partial x} = 0 \tag{116}
\]

and similarly

\[
\frac{\partial^2 v(x, y)}{\partial x^2} + \frac{\partial^2 v(x, y)}{\partial y^2} = -\frac{\partial}{\partial x} \frac{\partial u(x, y)}{\partial y} + \frac{\partial}{\partial y} \frac{\partial u(x, y)}{\partial x} = 0 \tag{117}
\]

This shows that both $u(x, y)$ and $v(x, y)$ satisfy the Laplace equation: $\Delta u = 0, \Delta v = 0$. Functions that satisfy the Laplace equation are called “harmonic functions”. $u(x, y)$ and $v(x, y)$ are called conjugate harmonic functions and they are not independent. From $u(x, y)$ one can obtain $v(x, y)$ up to a constant as

\[
v(x, y) = \int_{\gamma} \left( -\frac{\partial u(x, y)}{\partial y} dx + \frac{\partial u(x, y)}{\partial x} dy \right) \tag{118}
\]

where $\gamma$ is a path in $D$ from an arbitrary point $x_0, y_0$ to $x, y$. The integral in Eq. 118 is independent of the path since

\[
curl \left( -\frac{\partial u(x, y)}{\partial y}, \frac{\partial u(x, y)}{\partial x} \right) = \Delta u = 0 \tag{119}
\]

On the other hand the function $v(x, y)$ defined by Eq. 118 has

\[
\text{grad } v(x, y) \equiv \left( \frac{\partial v(x, y)}{\partial x}, \frac{\partial v(x, y)}{\partial y} \right) = \left( -\frac{\partial u(x, y)}{\partial y}, \frac{\partial u(x, y)}{\partial x} \right) \tag{120}
\]

and so $v$ and $u$ are related by the Cauchy-Riemann conditions.

Conformal mappings:
If $F(z)$ is an analytic function, the mapping

\[
z \to w = F(z) \tag{121}
\]
Figure 9: Illustration of a conformal mapping: the small square region centered at $z = 3.6 + 2.5i$ is transformed by $z \rightarrow z' = 0.3z^2 + 4 + 2i$. This figure as well as Figs. 10, 11, and 12 are computer generated with the OpenGL software. In all these figures the lines $\text{Re } z = \text{ constant}$ and $\text{Im } z = \text{ constant}$ are drawn in blue and purple, and the corresponding lines after the mapping are drawn in green and yellow.

defines a mapping of a region in the complex $z$ plane to a region in the complex $w$ plane. The mapping is “conformal” in the sense that angles are locally preserved. Consider indeed a neighborhood of $z$

$$z' = z + dz \quad (122)$$

with infinitesimal $dz$. Correspondingly we will have

$$w' = F(z') = F(z + dz) = F(z) + \frac{dF}{dz} \bigg|_z dz = w + dw \quad (123)$$

i.e.

$$dw = \frac{dF}{dz} \bigg|_z dz \quad (124)$$
Let
\[ dz = dr \, e^{i\phi}, \quad dw = ds \, e^{i\psi}, \quad \left. \frac{dF}{dz} \right|_z = Ae^{i\alpha} \] (125)

Equation 124 gives
\[ ds \, e^{i\psi} = Ae^{i\alpha} \, dr \, e^{i\phi} \] (126)

or
\[ ds = A \, dr \] (127)
\[ \psi = \phi + \alpha \] (128)

Equations 127, 128 show that the local transformation between the \( z \) and \( w \) planes is a uniform dilation by \( A \) and a rotation by \( \alpha \).

Figure 10: Illustration of the mapping of Eq. 136, which sends a rectangular region in the \( z \)-plane into a wedge shaped section of an annulus in the \( w \)-plane.

Let \( w = F(z) \), where \( F \) is an analytic function, define a mapping from a domain \( D \) in the complex \( z \) plane to a domain \( D' \) in the complex \( w \) plane. Denote by \( G(w) \) the inverse
mapping, so that

\[ G(F(z)) = z \quad (129) \]
\[ F(G(w)) = w \quad (130) \]

If \( f(z) \) is an analytic function in \( D \), then

\[ g(w) = f(G(w)) = f(z) \quad (131) \]

will be an analytic function in \( D' \). Then

\[ \text{Re } g(w) = \text{Re } f(z) \quad (132) \]
\[ \text{Im } g(w) = \text{Im } f(z) \quad (133) \]

will be harmonic functions in the \( w \) and \( z \) planes, respectively. Thus the mappings \( w = F(z) \), or \( z = G(w) \) will map harmonic functions into some different harmonic functions in the two planes. Sometime these mappings can help derive new harmonic functions, with interesting properties, from given ones.

Example:
Write \( z = x + iy \) and take as \( D \) the domain with

\[ a \leq x \leq b \quad (134) \]
\[ 0 \leq y \leq \alpha \quad (135) \]

Consider the analytic function \( f(z) = V_0(z - a)/(b - a) \). In electrostatics the trivially harmonic function \( \text{Re } f(z) = \text{Re } [V_0(z - a)/(b - a)] = V_0(x - a)/(b - a) \) can be seen as a potential function \( V \) taking values 0 for \( x = a \) and \( V_0 \) for \( x = b \). The lines \( y = c \) where the conjugate harmonic function \( y \) takes constant value equal to \( c \) are then the electric field lines.

Consider the mapping

\[ w \equiv re^{i\phi} = e^z = e^x e^{iy} \quad (136) \]

From this equation we can see that the domain \( D' \) in the complex \( w \) plane is delimited by the real axis, the ray emerging from the origin at an angle \( \alpha \) from the real axis, and the the two arcs of circle with center in the origin and radii \( R_0 = \exp(a), R_1 = \exp(b) \) respectively. In particular, if \( \alpha = 2\pi \), \( D' \) will cover the full annulus between \( R_0 \) and \( R_1 \). We have

\[ z = \log w \quad (137) \]

and thus in the \( w \) plane \( f(z) \) will be mapped to the function

\[ g(w) = \frac{V_0(\log w - a)}{b - a} \quad (138) \]
Figure 11: Illustration of the mapping of Eq. 136 with the rectangular region in the $z$-plane extended so that the corresponding region in the $w$-plane embraces the whole annulus. The green circles are now the equipotential lines of a charge placed at the origin, while the yellow lines are the lines of force of the electric field.

In this plane, with $w = r \exp(i\phi)$ the potential $V$ will be given by

$$V(w) = \text{Re} \left( \frac{V_0(\log r + i\phi - a)}{b - a} \right) = \left( \frac{V_0(\log r - a)}{b - a} \right)$$

(139)

with values which vary, of course, between 0 and $V_0$. The equipotential lines will be arcs of circle with center in the origin (or full circles if $\alpha = 2\pi$.) The conjugate function

$$\text{Im} \left( \frac{V_0(\log r + i\phi - a)}{b - a} \right) = \left( \frac{V_0\phi}{b - a} \right)$$

(140)

takes constant values along the rays emerging from the origin, which are the electric field lines of the above potential.
Finally, it is a useful exercise to consider the above potential in the full \( w \) plane and the two mappings \( w \rightarrow u \) defined by
\[
    w = (u - a)(u + a) \quad \text{(141)}
\]
or
\[
    w = \frac{u - a}{u + a} \quad \text{(142)}
\]
If we think of the potential in the \( w \) plane as due to a charge in the origin with an opposite charge at \( \infty \), the former mapping (Eq. 141) would map this potential into a potential with two equal charges at \( u = a \) and \( u = -a \), while the second mapping (Eq. 142) would map the potential into a potential with opposite charges at \( u = a \) and \( u = -a \).
Integral representations of analytic functions, exemplified by the Gamma function \( \Gamma(z) \)

Analytic functions may be defined by integral expressions. A notable example is the Gamma function:

\[
\Gamma(z) = \int_0^\infty x^{z-1} e^{-x} \, dx \quad \text{Re } z > 0 \tag{143}
\]

The integral converges for Re\(z > 0\). \( \Gamma(z) \) is defined throughout the complex \( z \) plane by analytic continuation. It has simple poles at all integers \( n \leq 0 \).

\[
\Gamma(1) = \int_0^\infty e^{-x} \, dx = 1 \tag{144}
\]

Integrating by parts one gets

\[
\Gamma(z + 1) = \int_0^\infty x^z e^{-x} \, dx = - \int_0^\infty x^z \frac{d e^{-x}}{d x} \, dx = z \int_0^\infty x^{z-1} e^{-x} \, dx = z \Gamma(z) \tag{145}
\]

By combining Eqs. 144 and 145 one gets

\[
\Gamma(n) = (n - 1)! \tag{146}
\]

for integer \( n > 0 \).

One can prove that \( \Gamma(z) \) has no zeroes. Therefore \( 1/\Gamma(z) \) is an entire function, i.e. is analytic in the whole complex plane. From the Taylor series expansion of the exponential function

\[
e^z = \sum_{n=0}^\infty c_n z^n = \sum_{n=0}^\infty \frac{1}{n!} z^n \tag{147}
\]

and the formula for the expansion coefficient

\[
c_n = \frac{1}{2\pi i} \oint z^{-(n+1)} e^z \, dz \tag{148}
\]

one gets

\[
\frac{1}{\Gamma(w)} = \frac{1}{(w - 1)!} = \frac{1}{2\pi i} \oint z^{-w} e^z \, dz \tag{149}
\]

for positive integer \( w \). This formula can be extended to the Hankel contour integral representation valid for all \( w \)

\[
\frac{1}{\Gamma(w)} = \frac{1}{(w - 1)!} = \frac{1}{2\pi i} \int_C z^{-w} e^z \, dz \tag{150}
\]

where the contour \( C \) starts at \(-\infty\), circles around the origin in counterclockwise direction and returns to \(-\infty\) (see Fig.13 ).

This formula can be used in conjunction with the “saddle point method” to calculate the asymptotic behavior of \( \frac{1}{\Gamma(w)} \) for \( w \to \infty \). The idea of the saddle point method is to find
Figure 13: Contour integral (shown in red) used in the Hankel contour integral representation of the function $1/\Gamma(w)$. For non integer $w$ the integrand has a branch point singularity and the contour is drawn around a cut (shown in green) along the negative imaginary axis. The illustration also shows in blue the contour of integration deformed to go through the saddle point in $P$.

A point in the complex plane where the derivative of the integrand vanishes, but is not a minimum or a maximum, rather it is a point, the saddle point, which is a maximum along some path $P$ and a minimum along the orthogonal path. By deforming the contour of integration so that it passes through the saddle point along $P$, if the behavior of the integrand becomes steeper and steeper as some variable goes to infinity, it will be possible to approximate the integral with a Gaussian integral and thus obtain the asymptotic behavior. It is clear from Eq. 150 that the integrand has a minimum along the positive real axis. Its value for $z = x > 0$ is

$$p(x) = x^{-w}e^x = e^{-w \log(x) + x}$$

with a minimum for

$$\frac{d[-w \log(x) + x]}{dx} = -\frac{w}{x} + 1 = 0$$

i.e. for

$$x = w$$
On the other hand if we set $z = w + iy$, i.e. if we look at the behavior of the integrand along a line through the minimum orthogonal to the real axis, we find

$$p(w + iy) = \exp[-w \log(w + iy) + w + iy] =$$

$$\exp\left[-w \log(w) - w \log \left(1 + \frac{iy}{w}\right) + w + iy\right] =$$

$$\exp\left[-w \log(w) - \left(iy - \frac{(iy)^2}{2w} + \ldots\right) + w + iy\right] =$$

$$\exp[-w \log(w) + w] \exp\left[-\frac{y^2}{2w} + \ldots\right]$$

which clearly shows that the integrand has a maximum along this path with Gaussian behavior. Thus, if we deform the contour $C$ to run parallel to the imaginary axis with $\text{Re } z = w$, we will be able to approximate $1/\Gamma(w)$ by

$$\frac{1}{\Gamma(w)} = \frac{1}{2\pi i} e^{-w \log(w) + w} \int_{-\infty}^{\infty} e^{-y^2/(2w)} i \, dy = \sqrt{\frac{w}{2\pi}} e^{-w \log(w) + w} = \sqrt{\frac{1}{2\pi}} e^{-(w-1/2) \log(w) + w}$$

(154)

and

$$\Gamma(w) = \sqrt{2\pi} e^{(w-1/2) \log(w) - w} = \sqrt{2\pi} w^{w-1/2} e^{-w}$$

(156)

This is the leading term in an expansion known as “Stirling’s formula”.

With $n! = \Gamma(n + 1)$ we also get

$$n! \sim \sqrt{2\pi} (n + 1)^{n+1/2} e^{-(n+1)} \sim \sqrt{2\pi n} \left(\frac{n}{e}\right)^n$$

(157)

neglecting subleading terms.