Cartesian coordinates.

\[ \vec{r} = x \hat{i} + y \hat{j} + z \hat{k} \]  

(1)

The unit vectors \( \hat{i}, \hat{j}, \hat{k} \) are orthonormal.

Change of frame. Example: rotate around the \( z \)-axis by \( \phi \) to a new frame \( \vec{i}', \vec{j}', \vec{k}' = \hat{k} \).

The new coordinates \( x', y', z' \) are given by

\[
\begin{pmatrix} x' \\ y' \\ z' \end{pmatrix} = R \begin{pmatrix} x \\ y \\ z \end{pmatrix}
\]  

(2)

with

\[
R = \begin{pmatrix} \cos \phi & \sin \phi & 0 \\ -\sin \phi & \cos \phi & 0 \\ 0 & 0 & 1 \end{pmatrix}
\]  

(3)

\( \vec{r} \) is unchanged:

\[ \vec{r} = x \hat{i} + y \hat{j} + z \hat{k} = x' \vec{i}' + y' \vec{j}' + z' \vec{k}' \]  

(4)
Sign check: with $\phi = 30^0$ as in Fig. 1, left panel, $x = 1, y = 0, z = 0$ becomes $x' = \cos \phi, y' = -\sin \phi, z' = 0$: $x' > 0, y' < 0$ o.k.

Note that if Eq. 2 is used instead as an operation on the vector, it describes the opposite rotation. If we keep the original basis, then

$$\vec{r} \rightarrow \vec{r}' = x' \hat{i} + y' \hat{j} + z' \hat{k}$$

and $\vec{r}$ rotates by $-\phi$ (check with the same matrix and $\phi = 30^0$ as above: now $\vec{r} = \hat{i} \rightarrow \vec{r}' = \cos \phi \hat{i} - \sin \phi \hat{j}$. See Fig. 1, right panel.

A general 3-d rotation can be obtained by a sequence of 2-d rotations around different axes (cfr. Euler angles.)

For a general rotation $R$, with

$$\vec{w} = R \vec{v}$$

we have

$$\vec{w} \cdot \vec{w} = (R \vec{v}) \cdot (R \vec{v}) = \vec{v} \cdot R^{tr} R \vec{v} = \vec{v} \cdot \vec{v}$$

hence

$$R^{tr} R = I, \quad \text{or} \quad R^{tr} = R^{-1}$$

$R$ is called orthogonal. With complex vectors conservation of the length requires

$$R^\dagger R \equiv (R^{tr})^* R = I, \quad \text{or} \quad R^\dagger = R^{-1}$$

and $R$ is called unitary.

Scalar and vector fields.

Scalar field: $F(x, y, z)$.

Vector field $\vec{v}(x, y, z) = (v_x(x, y, z), v_y(x, y, z), v_z(x, y, z))$.

Gradient:

$$\text{grad} F \equiv \nabla F = \left( \frac{\partial F}{\partial x}, \frac{\partial F}{\partial y}, \frac{\partial F}{\partial z} \right)$$

Nabla operator:

$$\nabla = \hat{i} \frac{\partial}{\partial x} + \hat{j} \frac{\partial}{\partial y} + \hat{k} \frac{\partial}{\partial z}$$

With $\vec{r} \rightarrow \vec{r} + d\vec{r}$:

$$dF = \nabla F \cdot d\vec{r}$$

Directional derivative:

$\vec{r} \equiv \vec{r}(s)$ (s is the arc length along the curve)

$$ds = |d\vec{r}|$$
\[ \frac{dF}{ds} = \nabla F \cdot \frac{d\vec{r}}{ds} \]  
(note: \( |d\vec{r}/ds| = 1 \))

With \( \gamma \equiv \vec{r}(s), \ \vec{r}(s_a) = \vec{r}_a, \ \vec{r}(s_b) = \vec{r}_b, \)

\[ \int_{\gamma} \nabla F \cdot d\vec{r} = \int_{s_a}^{s_b} \frac{dF}{ds} \, ds = F(\vec{r}_b) - F(\vec{r}_a) \]  
(15)

independently of \( \gamma \) (within a domain where \( F \) is regular and single valued.)

Divergence:

\[ \text{div} \ \vec{v} \equiv \nabla \cdot \vec{v} = \frac{\partial v_x}{\partial x} + \frac{\partial v_y}{\partial y} + \frac{\partial v_z}{\partial z} \]  
(16)

Laplace operator, or Laplacian

\[ \Delta \equiv \nabla^2 \equiv \nabla \cdot \nabla \equiv \nabla \cdot \text{div} \ \vec{v} \]  
(17)

Gauss’ theorem:

\[ \int_V \nabla \cdot \vec{v} \, dV \equiv \int_V \nabla \cdot \vec{v} \, dx \, dy \, dz = \int_\Sigma \vec{v} \cdot \hat{n} \, d\sigma \]  
(19)

where \( V \) is a closed volume bounded by the surface \( \Sigma \), \( \hat{n} \) is the outward normal to the surface, \( d\sigma \) is the area element of the surface, and \( \vec{v} \) is assumed to be regular inside \( V \).

Curl:

\[ \text{curl} \ \vec{v} \equiv \nabla \times \vec{v} = \left( \frac{\partial v_z}{\partial y} - \frac{\partial v_y}{\partial z}, \frac{\partial v_x}{\partial z} - \frac{\partial v_z}{\partial x}, \frac{\partial v_y}{\partial x} - \frac{\partial v_x}{\partial y} \right) \]  
(20)

Note:

\[ \nabla \cdot \nabla \times \vec{v} = 0 \]  
(21)

(div curl \( \vec{v} = 0 \))

Stokes’ theorem:

\( \Sigma \) is a surface bound by the oriented curve \( \gamma = \vec{r}(s), \ \hat{n} \) is the outward normal to the surface, \( d\sigma \) is the area element of the surface, \( s \) is the length along \( \gamma \), the orientation of \( \gamma \) is counterclockwise as seen from \( \hat{n}, \ \vec{v} \) is assumed to be regular in a domain enclosing \( \Sigma \), then:

\[ \int_{\gamma} \vec{v} \cdot \frac{d\vec{r}}{ds} \, ds = \int_{\Sigma} (\nabla \times \vec{v}) \cdot \hat{n} \, d\sigma \]  
(22)

Note: as a consequence of Eq. 21 and Gauss’ theorem, the integral in the r.h.s. of Eq. 22 is independent of the specific surface bounded by \( \gamma \), as it should.
If \( \text{curl} \ \vec{v} = 0 \) over a simply connected domain\(^1\), then the integral
\[
\Phi = \int_{\gamma} \vec{v} \cdot d\vec{r}
\] (23)
over any open curve \( \gamma \) from \( \vec{r}_a \) to \( \vec{r}_b \) within this domain depends only on the end points of \( \gamma \) but not on the path leading from \( \vec{r}_a \) to \( \vec{r}_b \)
\[
\Phi = \Phi(\vec{r}_a, \vec{r}_b)
\] (24)
In particular, if one takes a fixed \( \vec{r}_a = \vec{r}_0 \) and varies only \( \vec{r} = \vec{r}_b \), one obtains a “potential function” \( \Phi(\vec{r}) \) with the property that
\[
\vec{v} = \text{grad} \ \Phi(\vec{r})
\] (25)
The potential is only defined up to an additive constant (corresponding to different choices of the fixed point \( \vec{r}_a \)).\(^2\)

**Vector operators in 2-d**

(This will be important in the study of functions of complex variables.)

Consider a situation where there is no \( z \)-dependence and \( v_z = 0 \). Thus we effectively have a 2-d system:
\[
F = F(x, y)
\] (26)
\[
\vec{v} = (v_x(x, y), v_y(x, y))
\] (27)
Then:
\[
\nabla F = \left( \frac{\partial F}{\partial x}, \frac{\partial F}{\partial y} \right)
\] (28)
\[
\text{div} \ \vec{v} \equiv \nabla \cdot \vec{v} = \frac{\partial v_x}{\partial x} + \frac{\partial v_y}{\partial y}
\] (29)
Like \( \text{div} \ \vec{v} \), \( \text{curl} \ \vec{v} \) is also a scalar
\[
\text{curl} \ \vec{v} \equiv \nabla \times \vec{v} = \frac{\partial v_y}{\partial x} - \frac{\partial v_x}{\partial y}
\] (30)
If \( \text{curl} \ \vec{v} = 0 \) over a simply connected domain \( D \), then within \( D \) one can find a function \( \Phi(x, y) \) with
\[
v_x = \frac{\partial \Phi}{\partial x}
\] (31)
\[
v_y = \frac{\partial \Phi}{\partial y}
\] (32)

\(^1\)i.e. a domain where any closed curve can be smoothly contracted to a single point
\(^2\)Often the potential function is defined with a negative sign, for example the electric field is given by \( \vec{E} = -\text{grad} \ \Phi \), \( \Phi \) being the electrostatic potential.
If \( \operatorname{div} \vec{v} = 0 \), define

\[
\begin{align*}
    u_y &= v_x & (33) \\
    u_x &= -v_y & (34)
\end{align*}
\]

Then

\[ \operatorname{curl} \vec{u} = \frac{\partial u_y}{\partial x} - \frac{\partial u_x}{\partial y} = \frac{\partial v_x}{\partial x} + \frac{\partial v_y}{\partial y} = 0 \]  (35)

and one can find a function \( \Psi(x,y) \) with

\[
\begin{align*}
    v_x &= u_y = \frac{\partial \Psi}{\partial y} & (36) \\
    v_y &= -u_x = -\frac{\partial \Psi}{\partial x} & (37)
\end{align*}
\]

If both \( \operatorname{curl} \vec{v} = 0 \) and \( \operatorname{div} \vec{v} = 0 \), then \( \Phi \) and \( \Psi \) satisfy the Laplace equation

\[
\begin{align*}
    \Delta \Phi &= 0 & (38) \\
    \Delta \Psi &= 0 & (39)
\end{align*}
\]

**Curvilinear coordinates**

Exemplified by spherical coordinates \( r, \theta, \phi \):

\[
\begin{align*}
    x &= r \sin \theta \cos \phi \\
    y &= r \sin \theta \sin \phi \\
    z &= r \cos \theta
\end{align*}
\]

\[ \vec{r} = \hat{r}(r, \theta, \phi) \]

\[
d\vec{r} = \frac{\partial \vec{r}}{\partial r} dr + \frac{\partial \vec{r}}{\partial \theta} d\theta + \frac{\partial \vec{r}}{\partial \phi} d\phi = h_r \hat{r} dr + h_\theta \hat{\theta} d\theta + h_\phi \hat{\phi} d\phi \]  (43)

with

\[
\begin{align*}
    h_r &= \left| \frac{\partial \vec{r}}{\partial r} \right| & (44) \\
    h_\theta &= \left| \frac{\partial \vec{r}}{\partial \theta} \right| & (45) \\
    h_\phi &= \left| \frac{\partial \vec{r}}{\partial \phi} \right| & (46)
\end{align*}
\]

The three vectors \( \hat{r}, \hat{\theta}, \hat{\phi} \) are orthonormal: spherical coordinates form a set of orthogonal coordinates.
\[ F = F(r, \theta, \phi) \]. Calculate \( \text{grad} \ F \):

\[
dF = \text{grad} \ F \cdot d\vec{r} = \text{grad} \ F \cdot (h_r \hat{r} \, dr + h_\theta \hat{\theta} \, d\theta + h_\phi \hat{\phi} \, d\phi)
\]  

(47)

but also

\[
dF = \frac{\partial F}{\partial r} \, dr + \frac{\partial F}{\partial \theta} \, d\theta + \frac{\partial F}{\partial \phi} \, d\phi
\]  

(48)

The differentials \( dr, d\theta, d\phi \) are independent, thus we will have

\[
h_r \text{ grad } F \cdot \hat{r} = \frac{\partial F}{\partial r}
\]  

(49)

\[
h_\theta \text{ grad } F \cdot \hat{\theta} = \frac{\partial F}{\partial \theta}
\]  

(50)

\[
h_\phi \text{ grad } F \cdot \hat{\phi} = \frac{\partial F}{\partial \phi}
\]  

(51)

The components of \( \text{grad} \ F \) along the three orthogonal directions are thus \( (1/h_r)(\partial F/\partial r) \), \( (1/h_\theta)(\partial F/\partial \theta) \), \( (1/h_\phi)(\partial F/\partial \phi) \) and we get

\[
\text{grad } F = \frac{1}{h_r} \frac{\partial F}{\partial r} \hat{r} + \frac{1}{h_\theta} \frac{\partial F}{\partial \theta} \hat{\theta} + \frac{1}{h_\phi} \frac{\partial F}{\partial \phi} \hat{\phi}
\]  

(52)

With spherical coordinates \( h_r = 1, h_\theta = r, h_\phi = r \sin \theta \) and

\[
\text{grad } F = \frac{\partial F}{\partial r} \hat{r} + \frac{1}{r} \frac{\partial F}{\partial \theta} \hat{\theta} + \frac{1}{r \sin \theta} \frac{\partial F}{\partial \phi} \hat{\phi}
\]  

(53)

\[ \vec{v} = v_r(r, \theta, \phi) \hat{r} + v_\theta(r, \theta, \phi) \hat{\theta} + v_\phi(r, \theta, \phi) \hat{\phi} \]. Calculate \( \text{div} \ \vec{v} \):

\[
\Phi_{\hat{r}} = h_\theta h_\phi v_r \, d\theta \, d\phi
\]  

(54)

![Figure 2: Flux of a vector \( \vec{v} \) through an infinitesimal cube with sides along the coordinate lines.](image-url)
As \( r \) increases by \( dr \) along the \( \hat{r} \) side, the differential of this flux, i.e. the net outgoing flux will be

\[
d\Phi_{\hat{r}} = \frac{\partial(h_\theta h_\phi v_r)}{\partial r} \, dr \, d\theta \, d\phi
\]  

(55)

Adding similar contributions through the \( \hat{\phi}, \hat{r} \) and \( \hat{\theta} \) faces we find a total outgoing flux

\[
d\Phi = \left( \frac{\partial(h_\theta h_\phi v_r)}{\partial r} + \frac{\partial(h_\phi h_r v_\theta)}{\partial \theta} + \frac{\partial(h_r h_\theta v_\phi)}{\partial \phi} \right) dr \, d\theta \, d\phi
\]  

(56)

On the other hand by Gauss’ theorem \( d\Phi \) must be equal to the integral of \( \text{div} \, \vec{v} \) over the cube, i.e.

\[
d\Phi = h_r h_\theta h_\phi \left( \text{div} \, \vec{v} \right) dr \, d\theta \, d\phi
\]  

(57)

Comparing Eqs. 56, 57 we get

\[
\text{div} \, \vec{v} = \frac{1}{h_r h_\theta h_\phi} \left( \frac{\partial(h_\theta h_\phi v_r)}{\partial r} + \frac{\partial(h_\phi h_r v_\theta)}{\partial \theta} + \frac{\partial(h_r h_\theta v_\phi)}{\partial \phi} \right)
\]  

(58)

Specifically

\[
\text{div} \, \vec{v} = \frac{1}{r^2 \sin \theta} \left( \frac{\partial(r^2 \sin \theta v_r)}{\partial r} + \frac{\partial(r \sin \theta v_\theta)}{\partial \theta} + \frac{\partial(r v_\phi)}{\partial \phi} \right) = \frac{\partial v_r}{\partial r} + \frac{2}{r} \, v_r + \frac{1}{r} \frac{\partial v_\theta}{\partial \theta} + \cot \theta \, v_\theta + \frac{1}{r} \frac{\partial v_\phi}{\partial \phi}
\]  

(59)

curl \( \vec{v} \):

Proceeding in a similar fashion from Stokes’ theorem one finds

\[
\text{curl} \, \vec{v} = \frac{1}{r^2 \sin \theta} \left( \frac{\partial(r \sin \theta u_\phi)}{\partial \theta} - \frac{\partial(r u_\theta)}{\partial \phi} \right) \hat{r} + \frac{1}{r \sin \theta} \left( \frac{\partial u_r}{\partial \phi} - \frac{\partial(r \sin \theta u_\theta)}{\partial r} \right) \hat{\theta} + \frac{1}{r} \left( \frac{\partial(r u_\theta)}{\partial r} - \frac{\partial u_r}{\partial \theta} \right) \hat{\phi}
\]  

(60)

The Laplacian:

\[
(\frac{\partial}{\partial r} + \frac{2}{r}) \left( \frac{\partial F}{\partial r} \right) + \left( \frac{1}{r} \frac{\partial}{\partial \theta} + \cot \theta \right) \left( \frac{1}{r} \frac{\partial F}{\partial \theta} \right) + \left( \frac{1}{r \sin \theta} \frac{\partial}{\partial \phi} \right) \left( \frac{1}{r \sin \theta} \frac{\partial F}{\partial \phi} \right) = \frac{1}{r^2} \frac{\partial}{\partial r} \left( r^2 \frac{\partial F}{\partial r} \right) + \frac{1}{r^2 \sin \theta} \frac{\partial}{\partial \theta} \left( \sin \theta \frac{\partial F}{\partial \theta} \right) + \frac{1}{r^2 \sin^2 \theta} \frac{\partial^2 F}{\partial \phi^2}
\]  

(61)