PY 501 - Mathematical Physics
Assignment 5 - October 17, 2021
To be completed and returned in class on Monday, October 25, 2021.

Do all four problems. Each correct solution will be given 25 points.

Problem 1:
Establish the relation between the evolution equations 5 and 6 in the lecture notes “Nonlinear, ordinary differential equations” and Hamiltonian mechanics, by identifying $x$ as the coordinate of a particle (i.e. a point-like object) and $y$ as its momentum $p$. We will continue using $p$ rather than $y$. For your convenience we reproduce those equations here:

$$\frac{dp}{dt} = F(p, x) \quad (1)$$
$$\frac{dx}{dt} = G(p, x) \quad (2)$$

A) Assume that the kinetic energy of the particle is $p^2/(2m)$, where $m$ is the particle’s mass, and that its potential energy is $V(x)$. Write down the equations of motion and identify the functions $F(p, x)$ and $G(p, x)$. (Do not introduce the Hamiltonian as yet.)

B) Introduce a two component vector $\vec{v}(p, x)$ in the $x, p$-plane with components

$$v_1(p, x) = G(p, x) \quad (3)$$
$$v_2(p, x) = -F(p, x) \quad (4)$$

(Note the sign in Eq. 4.) Show that curl $\vec{v} = \partial v_2/\partial p - \partial v_1/\partial x = 0$.

C) From B) it follows that there exist a function $H(p, x)$ with

$$\vec{v}(p, x) = \text{grad} \; H(p, x) \quad (5)$$

Find the function $H(p, x)$ denoting the arbitrary constant of integration by $V_0$. 

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D) Prove that $H$ is constant along the motion of the particle.

**Solution**

A) The equations of motion of the particle are

\[
\begin{align*}
\frac{dp}{dt} &= -\frac{dV(x)}{dx} \\
\frac{dx}{dt} &= \frac{p}{m}
\end{align*}
\]  

(6) (7)

where $-dV/dx$ is the force acting on the particle. From these equations we get

\[
\begin{align*}
F(p,x) &= -\frac{dV(x)}{dx} \\
G(p,x) &= \frac{p}{m}
\end{align*}
\]  

(8) (9)

B) The component of $\vec{v}$ are

\[
\begin{align*}
v_1(p,x) &= \frac{p}{m} \\
v_2(p,x) &= \frac{dV(x)}{dx}
\end{align*}
\]  

(10) (11)

The fact that $\partial v_2/\partial p - \partial v_1/\partial x = 0$ is obvious.

C) We find $H(p,x)$ by integrating first $v_1$ along the $p$-axis from 0 to $p$

\[
\int_0^p \frac{p'}{m} \, dp' = \frac{p^2}{2m}
\]  

(12)

We integrate now $v_2$ on a parallel to the $p$-axis at $p$ from 0 to $x$ (the value of $p$ is actually immaterial since $p$ does not appear in $v_2$)

\[
\int_0^x \frac{dV(x')}{dx} \, dx' = V(x) - V_0
\]  

(13)

We can incorporate $V_0$ into the definition of $V(x)$ and will neglect it in what follows. Adding the two integrals we find

\[
H(p,x) = \frac{p^2}{2m} + V(x)
\]  

(14)
D) We have
\[
\frac{dH(p,x)}{dt} = \frac{\partial H}{\partial p} \frac{dp}{dt} + \frac{\partial H}{\partial x} \frac{dx}{dt} = v_1(p,x)[-v_2(p,x)] + v_2(p,x)v_1(p,x) = 0 \quad (15)
\]
which shows that the value of \( H \) is indeed constant along the motion.

Problem 2:
Consider the generalization of the Lotka-Volterra equations, which, after a rescaling of the variables and time, takes the form
\[
\frac{dx(t)}{dt} = x(t) - cx(t)^2 - x(t)y(t) \quad (16)
\]
\[
\frac{dy(t)}{dt} = -ay(t) + b + x(t)y(t) \quad (17)
\]
As in the original Lotka-Volterra equations the variables \( x \) and \( y \), which should be positive, represent prey and predator. \( a \) is a positive constant, \( b < a \) and \( c \) are non-negative constants.

A) The equations have a fixed-point with \( x = 0, y = b/a \), and another fixed-point at \( x = x_0, y = y_0 \) with \( x_0 \neq 0 \). Find the values of \( x_0, y_0 \) for this second fixed-point.

B) Linearize the evolution equations 16 and 17 around the fixed-point you found in part A, by setting \( x = x_0 + \delta x, y = y_0 + \delta y \) and expanding to first order in \( \delta x, \delta y \). Keep the symbols \( x_0, y_0 \) in the linearized equations; do not replace them with the solution you found in part A, which would complicate the algebra.

C) Use the linearized equations you obtained in part B to decide whether the fixed-point is attractive, stable, or repulsive with the following values of the parameters:
1) \( a = 2, b = c = 0 \)
2) \( a = 2, b = 1/2, c = 1/2 \)

You will need the values of \( x_0, y_0 \) corresponding to these parameters. You may use the solution obtained in part A) for general values of \( a, b, c \) or solve the equations for the fixed-point with these specific parameters.

(By stable we mean that in an infinitesimal neighborhood of the fixed-point the trajectories form closed loops.)
Note: If the linearized equations are written in matrix form as
\[
\begin{pmatrix}
\frac{d \delta x}{dt} \\
\frac{d \delta y}{dt}
\end{pmatrix} = M \begin{pmatrix}
\delta x \\
\delta y
\end{pmatrix}
\] (18)
the behavior of their solution in the neighborhood of the fixed-point is determined by the eigenvalues of \( M \)
\[
\lambda_{1,2} = \alpha_{1,2} + i\beta_{1,2}
\] (19)
The corresponding eigenvector components \( \delta \vec{v}_i \) will evolve according to
\[
\delta \vec{v}_i(t) = \delta \vec{v}_i(0) e^{\alpha_i t} e^{i\beta_i t}
\] (20)
and the behavior of the solution in the neighborhood of the fixed-point may be read off these equations.

Solution
A) By demanding that the time derivatives in Eqs. 16, 17 vanish we obtain the equations
\[
x_0 - cx_0^2 - x_0 y_0 = 0 \quad (21)
\]
\[
-ay_0 + b + x_0 y_0 = 0 \quad (22)
\]
By dividing Eq. 21 by \( x_0 \) we obtain
\[
1 - cx_0 - y_0 = 0
\] (23)
i.e.
\[
y_0 = 1 - cx_0
\] (24)
By substituting into Eq. 22 we obtain the equation for \( x_0 \)
\[
-a + ac x_0 + b + x_0 - cx_0^2 = -cx_0^2 + (1 + ac)x_0 - (a - b) = 0
\] (25)
with solution
\[
x_0 = \frac{2(a - b)}{1 + ac \pm \sqrt{(1 + ac)^2 - 4(a - b)c}} = \frac{2(a - b)}{1 + ac \pm \sqrt{(1 - ac)^2 + 4bc}}
\] (26)
We discard the solution with the negative sign in the denominator, which leads to negative \( x_0 \), and find

\[
x_0 = \frac{2(a - b)}{1 + ac + \sqrt{(1 - ac)^2 + 4bc}} \quad (27)
\]

\[
y_0 = 1 - cx_0 = \frac{1 - ac + \sqrt{(1 - ac)^2 + 4bc} + 2bc}{1 + ac + \sqrt{(ac - 1)^2 + 4bc}} \quad (28)
\]

B) By replacing \( x, y \) with \( x_0 + \delta x, y_0 + \delta y \) in Eqs. 16, 17 and expanding to first order we find

\[
\frac{d\delta x}{dt} = \delta x - 2cx_0\delta x - y_0\delta x - x_0\delta y \quad (29)
\]

\[
\frac{d\delta y}{dt} = y_0\delta x - a\delta y + x_0\delta y \quad (30)
\]

from which we read the form of the matrix \( M \)

\[
M = \begin{pmatrix} 1 - 2cx_0 - y_0 & -x_0 \\ y_0 & -a + x_0 \end{pmatrix} \quad (31)
\]

which we will use in part C.

C 1) With \( a = 2, b = c = 0 \) we get first

\[
x_0 = 2 \\
y_0 = 1 \quad (32)
\]

and then

\[
M = \begin{pmatrix} 0 & -2 \\ 1 & 0 \end{pmatrix} \quad (33)
\]

The eigenvalues of \( M \) are \( \lambda_{\pm} = \pm i\sqrt{2} \) and thus we deduce

\[
\delta \vec{v}_{\pm}(t) = \delta \vec{v}_{\pm}(0) e^{\pm \beta t} \quad (34)
\]

which means that the fixed-point is stable.

2) With \( a = 2, b = 1/2, c = 1/2 \) we get first

\[
x_0 = 1 \\
y_0 = 1/2 \quad (35)
\]

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and then
\[
M = \begin{pmatrix}
-1/2 & -1 \\
1/2 & -1
\end{pmatrix}
\]  
(36)

The eigenvalues of \( M \) are obtained by solving
\[
\det(M - \lambda I) = (-1/2 - \lambda)(-1 - \lambda) + 1/2 = \lambda^2 + 3\lambda/2 + 1 = 0
\]  
(37)

and are given by
\[
\lambda_{\pm} = \alpha + i\beta = -3/2 \pm \frac{\sqrt{9/4 - 4}}{2} = -\frac{3}{4} \pm i\frac{\sqrt{7}}{4}
\]  
(38)

With \( \alpha_{\pm} = -3/4 \) we conclude that the fixed-point is attractive.

**Problem 3:**

Consider the set of coupled nonlinear equations for the evolution in time of the variables \( x(t) \) and \( y(t) \):
\[
\frac{dx}{dt} = y^3 - 2xy + y
\]  
(39)
\[
\frac{dy}{dt} = -x^3 + x^2 + y^2 - x
\]  
(40)

A) Verify that the equations have the following fixed points
\[
P_0 : \quad x = 0 \quad y = 0
\]  
(41)
\[
P_1 : \quad x = 1 \quad y = 1
\]  
(42)
\[
P_2 : \quad x = 1 \quad y = -1
\]  
(43)

B) Linearize the equations in the neighborhoods of all the three fixed points.

C) From the linearized equations find whether the fixed points fixed point are attractive, stable, or repulsive. (Note, the treatment of \( P_1 \) and \( P_2 \) requires special care.)

D) Define the two-dimensional vector field in the \( x-y \) plane with components
\[
v_1(x, y) = x^3 - x^2 - y^2 + x
\]  
(44)
\[
v_2(x, y) = y^3 - 2xy + y
\]  
(45)

Calculate
\[
\text{curl} \ \vec{v} = \frac{\partial v_2}{\partial x} - \frac{\partial v_1}{\partial y}
\]  
(46)
and show that curl $\vec{v}$ vanishes.

E) From the result in D) it follows that there exists a function $H(x, y)$ with the property that

$$\vec{v} = \text{grad} \, H(x, y)$$

Find the function $H(x, y)$ assuming that its value is zero for $x = y = 0$ (in order to fix the constant in the definition of $H$.)

F) Calculate the time dependence of $H[x(t), y(t)]$ and show that $H$ is a constant of the motion. (You will need $dx/dt = v_2(x, t)$, $dy/dt = -v_1(x, t)$, see Eqs. 39, 40 at the beginning of this problem.)

Solution

A) We easily calculate

$$y^3 - 2xy + y = 0 \text{ with } x = 0, y = 0; x = 1, y = 1; x = 1, y = -1 \quad (48)$$

$$-x^3 + x^2 + y^2 - x = 0 \text{ with } x = 0, y = 0; x = 1, y = 1; x = 1, y = -1 \quad (49)$$

B) By replacing $x, y$ with $x_0 + \delta x, y_0 + \delta y$ in Eqs. 39, 40 and expanding to first order we find

$$\frac{d\delta x}{dt} = 3y^2\delta y - 2y\delta x - 2x\delta y + \delta y$$

$$\frac{d\delta y}{dt} = -3x^2\delta x + 2x\delta x + 2y\delta y - \delta x$$

C) We write Eqs. 50 51 in matrix form as

$$\begin{pmatrix} \frac{d\delta x}{dt} \\ \frac{d\delta x}{dt} \end{pmatrix} = M \begin{pmatrix} \delta x \\ \delta y \end{pmatrix}$$

with

$$M = \begin{pmatrix} -2y & 3y^2 - 2x + 1 \\ -3x^2 + 2x - 1 & 2y \end{pmatrix}$$

At the fixed point $P_0$ we have

$$M = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}$$

as in the harmonic oscillator motion and $P_0$ is stable.
At the fixed point $P_1$ we have

$$M = \begin{pmatrix} -2 & 2 \\ -2 & 2 \end{pmatrix}$$  \hspace{1cm} (55)$$

This matrix has zero determinant. We make sense of what happens at this fixed point by writing the equations for $\delta x, \delta y$ explicitly

$$\frac{d \delta x}{dt} = -2\delta x + 2\delta y$$  \hspace{1cm} (56)$$

$$\frac{d \delta y}{dt} = -2\delta x + 2\delta y$$  \hspace{1cm} (57)$$

if we change variables to $\delta u = (\delta x + \delta y)/2$, $\delta v = \delta x - \delta y$, adding and subtracting the equations we find

$$\frac{d \delta u}{dt} = -2\delta v$$  \hspace{1cm} (58)$$

$$\frac{d \delta v}{dt} = 0$$  \hspace{1cm} (59)$$

This means that if, at some $t = 0$, we displace these variables by an infinitesimal amount from the fixed point, the value of $\delta v$ will remain stationary at $\delta v(t) = \delta v(0)$, while $\delta u$ will grow as

$$\delta u(t) = \delta u(0) + \delta v(0) t$$  \hspace{1cm} (60)$$

In terms of the original variables, what happens is that their difference will stay constant while their mean value will move away from the fixed point value. The resulting behavior is cusp-like, as is also put in evidence by a numerical solution of the equations of motion.

About $P_2$ we notice that the equations of motion are invariant under the transformation $y \to -y, t \to -t$, which exchanges $P_1$ and $P_2$. So the properties of $P_2$ will be identical to those of $P_1$.

D) We have

$$\frac{\partial v_2}{\partial x} = -2y$$  \hspace{1cm} (61)$$

$$\frac{\partial v_1}{\partial y} = -2y$$  \hspace{1cm} (62)$$
Figure 1: Phase-space trajectories of four solutions to Eqs. 39, 40. The solutions shown in red passes very close to the fixed points $P_1$ and $P_2$ and illustrates the cusp-like behavior of the trajectories there.

from which it follows that curl $\vec{v} = 0$.

E) We integrate $v_1$ along the $x$-axis from 0 to $x$ to find

$$\int_0^x v_1(x',0) \, dx' = \int_0^x (x'^3 - x'^2 + x') \, dx' = \frac{x^4}{4} - \frac{x^3}{3} + \frac{x^2}{2} \quad (63)$$

We integrate then $v_2$ along the line at $x$ parallel to the $y$-axis from 0 to $y$ to find

$$\int_0^y v_2(x,y') \, dy' = \int_0^y (y'^3 - 2xy' + y') \, dy' = \frac{y^4}{4} - xy^2 + \frac{y^2}{2} \quad (64)$$

Combining the two we find

$$H(x,y) = \int_0^x v_1(x',0) \, dx' + \int_0^y v_2(x,y') \, dy' = \frac{x^4}{4} + \frac{y^4}{4} - \frac{x^3}{3} - xy^2 + \frac{x^2}{2} + \frac{y^2}{2} \quad (65)$$

F) We have

$$\frac{dH(x,y)}{dt} = \frac{\partial H}{\partial x} \frac{dx}{dt} + \frac{\partial H}{\partial y} \frac{dy}{dt} = v_1(x,t)v_2(x,y) - v_2(x,t)v_1(x,y) = 0 \quad (66)$$
It follows that the trajectories are the lines of constant $H$.

Figure 1 illustrates the phase-space trajectory of a few solutions of the system of equations considered in this problem.

**Problem 4:**

Solve Eq. 62 in Lecture 4, which we reproduce here (without the tilde over $x$ for convenience of notation)

$$\frac{d^2 x(t)}{dt^2} + x(t) = -\left[R - \frac{R^3}{4}\right] \sin t + \frac{R^3}{4} \sin 3t$$

(67)

looking for a solution which vanishes for $t \leq 0$.

Show that if $R = 2$ the magnitude of the solution will remain bound, while if $R \neq 2$ the solution will contain a term which grows with $t$.

Note: The solution will show not only that $x(t)$ grows in magnitude when $R \neq 2$, but, more specifically, that the full approximation (see Eqs. 52 and 53 in Lecture 4)

$$x(t) = R \cos t + \epsilon \tilde{x}(t)$$

(68)

$$y(t) = -R \sin t + \epsilon \frac{d \tilde{x}(t)}{dt}$$

(69)

evolves inward, toward the $R = 2$ limiting cycle, if $R > 2$, while it evolves outward, again toward the limiting cycle, if $R < 2$. Figure 2 shows a comparison between the approximate and exact trajectories for $R = 1, 2$ and 3.

**Solution**

We want to find the solution to the equation

$$\frac{d^2 x(t)}{dt^2} + x(t) = F(t)$$

(70)

where

$$F(t) = -\left[R - \frac{R^3}{4}\right] \sin t + \frac{R^3}{4} \sin 3t$$

(71)

or

$$F(t) = -\alpha \sin t + \beta \sin 3t$$

(72)
Figure 2: Comparison of exact phase-space trajectories, shown in green, with the approximate trajectories of Eqs. 68, 69, shown in red, all with $\epsilon = 0.02$. The initial data were $x = R, y = 0$ for all the trajectories, with $R = 1, 2, 3$. With $x$ and $y$ on the horizontal and vertical axes the trajectories evolve clockwise. The evolution is followed up to $t = 4\pi$ (two cycles.) In the figure the approximate trajectories are overlaid on the exact ones, so the exact trajectories are not visible when the difference between the two is less than the width of the lines, which happens for the entire trajectories with $R = 1$ and 2. It is apparent that the term growing with $t$ in the solution to Eq. 67 causes the approximate trajectories evolve inward for $R > 2$ and outward for $R < 2$. The exact trajectories have been obtained with a fourth-order Runge-Kutta numerical integration of the equations of motion.

with

\[
\alpha = \left[ R - \frac{R^3}{4} \right] \quad (73)
\]

\[
\beta = \frac{R^3}{4} \quad (74)
\]
We will solve with
\[ F(t) = -\alpha e^{it} + \beta e^{3it} \] (75)
and take the imaginary part at the end of the calculations.

We use the method of variation of constants and take
\[ x_1(t) = e^{it} \] (76)
\[ x_2(t) = e^{-it} \] (77)
as the two solutions of the homogeneous equation. Their Wronskian is
\[ W(t) = -2i \] (78)

Following the method of variation of the constants the solution will be given by
\[ x(t) = c_1(t)y_1(t) + c_2(t)y_2(t) \] (79)

with
\[ c_1(t) = -\int_0^t F(t')x_2(t') \, dt' \] (80)
\[ c_2(t) = \int_0^t F(t')x_1(t') \, dt' \] (81)

Inserting these results into Eq. 79 we get
\[ x(t) = \frac{\alpha t e^{it}}{2} - \frac{\beta}{4} (e^{3it} - e^{it}) - \frac{\alpha}{4} (e^{it} - e^{-it}) + \frac{\beta}{8} (e^{3it} - e^{-it}) \] (82)

By taking the imaginary part of the r.h.s. we find our actual solution
\[ x(t) = \alpha t \cos t - \sin t + \beta \frac{3 \sin t - \sin 3t}{8} = \left[R - \frac{R^3}{4}\right] \frac{t \cos t - \sin t}{2} + R^3 \frac{3 \sin t - \sin 3t}{32} \] (83)

This result clearly shows that for \( R = 2 \) the term of order \( \epsilon \) in the solution will remain bound, while it will exhibit an oscillatory component of magnitude increasing with \( t \) if \( R \neq 2 \).