PY 501 - Mathematical Physics

Assignment 10 - November 21, 2021.

To be completed and returned in class on Wednesday, December 1, 2021.

Do all four problems. Each correct solution will be given 25 points, with points taken away for errors (or for absence of explanation) according to the severity of the error.

Problem 1:
A) Find the characteristics of the differential equation
\[
\frac{\partial^2 \phi(x,t)}{\partial t^2} - v(x)^2 \frac{\partial^2 \phi(x,t)}{\partial x^2} = 0
\] (1)
where
\[
v(x) = c \frac{x^2 + \ell^2}{x^2 + 2\ell^2}
\] (2)

*Hint:* The characteristics are the trajectories of a particle moving with velocity \(\pm v(x)\).

B) Consider a perturbation at \(x = t = 0\) which propagates toward positive \(x\) according to Eq. 1. How far behind would it lag for \(t \to \infty\) with respect to a ray of light also emitted at \(x = t = 0\) and propagating in the positive \(x\) direction with constant velocity \(c\)?

C) Draw a space-time diagram of the trajectories of the perturbation and of the ray of light with \(c = 1\) and \(\ell = 1\). (Place the \(t\)-axis in vertically in the drawing.)

Solution
A) The equation for the characteristics is
\[
\frac{dx(t)}{dt} = \pm v(x) = \pm c \frac{x^2 + \ell^2}{x^2 + 2\ell^2}
\] (3)

By integrating we get
\[
\pm \int_0^x \frac{x'^2 + 2\ell^2}{x'^2 + \ell^2} dx' = c(t - t_0)
\] (4)
where \(t_0\) denotes the time at which the characteristic crosses the time axis. We have
\[
\int_0^x \frac{x'^2 + 2\ell^2}{x'^2 + \ell^2} dx = x + \ell \int_0^x \frac{dx'}{x'^2 + \ell^2} = x + \ell \arctan \frac{x}{\ell}
\] (5)

so the characteristics are given by the equations
\[
x + \ell \arctan \frac{x}{\ell} \pm c(t - t_0) = 0
\] (6)
B) The equation of the characteristic going through the origin and evolving in the positive $x$ direction is

$$x + \ell \arctan \frac{x}{\ell} = ct$$  \hspace{1cm} (7)

while the equation describing a ray of light is

$$x = ct$$  \hspace{1cm} (8)

We see that the time it takes for the perturbation to arrive at a definite $x$ is larger by \((\ell/c) \arctan \frac{x}{\ell}\) than the time \(x/c\) it takes to the light ray to get there. In the limit \(t \to \infty, x \to \infty\) this time lag will tend to \((\ell/c) \arctan \infty = \pi \ell/(2c)\). Since both the perturbation and the light ray will be propagating there with the speed of light, the corresponding distance will be \(\pi \ell/2\).

C) Figure 1 shows the trajectories of the perturbation, which moves along a characteristic of the hyperbolic equations, and of the ray of light, with \(c = 1\) and \(\ell = 1\).

Figure 1: Illustration of the solution to problem 1.
Problem 2:

This problem and problem two deal with the separation of variables in parabolic coordinates for a quantum system with Coulomb potential. They draw from “Quantum Mechanics-Nonrelativistic Theory” by L. D. Landau and E. M. Lifshitz.

In three-dimensional space introduce three curvilinear coordinates \( \xi, \eta, \phi \), called “parabolic coordinates” related to the Cartesian coordinates by

\[
\begin{align*}
    x &= \sqrt{\xi \eta} \cos \phi \\
    y &= \sqrt{\xi \eta} \sin \phi \\
    z &= \frac{1}{2} (\xi - \eta) \\
    r &= \sqrt{x^2 + y^2 + z^2} = \frac{1}{2} (\xi + \eta)
\end{align*}
\]

with \( \xi, \eta \leq \infty \); or, conversely,

\[
\begin{align*}
    \xi &= r + z \\
    \eta &= r - z \\
    \phi &= \phi = \arctan \frac{y}{x}
\end{align*}
\]

A) The following is true for all \( \phi = \) constant planes, but to be specific consider the \( \phi = 0 \) plane, i.e. the \( xz \)-plane with \( x \geq 0 \). Show that the coordinate lines \( \xi = \) constant and \( \eta = \) constant are parabolae (more precisely one of the two branches of a parabola) which meet at right angles. Plot or draw the coordinate lines with \( \xi \) and \( \eta \) equal to 5, 10, 15, 20.

B) Calculate the expression of the Laplacian in parabolic coordinates. As a check you should find that it is given by

\[
\Delta = \frac{4}{\xi + \eta} \left[ \frac{\partial}{\partial \xi} \left( \xi \frac{\partial}{\partial \xi} \right) + \frac{\partial}{\partial \eta} \left( \eta \frac{\partial}{\partial \eta} \right) \right] + \frac{1}{\xi \eta} \frac{\partial^2}{\partial \phi^2}
\]

\[
\Delta = \frac{1}{\rho} \frac{\partial}{\partial \rho} \rho \frac{\partial}{\partial \rho} + \frac{\partial^2}{\partial z^2} + \frac{1}{\rho^2} \frac{\partial^2}{\partial \phi^2}
\]

\[\Delta = \frac{4}{\xi + \eta} \left[ \frac{\partial}{\partial \xi} \left( \xi \frac{\partial}{\partial \xi} \right) + \frac{\partial}{\partial \eta} \left( \eta \frac{\partial}{\partial \eta} \right) \right] + \frac{1}{\xi \eta} \frac{\partial^2}{\partial \phi^2}
\]

\[\Delta = \frac{1}{\rho} \frac{\partial}{\partial \rho} \rho \frac{\partial}{\partial \rho} + \frac{\partial^2}{\partial z^2} + \frac{1}{\rho^2} \frac{\partial^2}{\partial \phi^2}
\]

Hint: I found it convenient to start from the expression of the Laplacian in cylindrical coordinates

changing then coordinates to

\[
\begin{align*}
    \xi &= \sqrt{\rho^2 + z^2} + z \\
    \eta &= \sqrt{\rho^2 + z^2} - z
\end{align*}
\]

where \( \rho = \sqrt{r^2 - z^2} \). The algebra is demanding and must be done with care, but it is manageable.
Solution

A) The points with coordinates satisfying $r = z + \eta$ are equidistant from the origin and the line $z = -\eta$. The locus of points with this property is a parabola with focus in the origin. The fact that the $\eta = \text{constant}$ lines are parabolae can also be seen algebraically. From $\eta = r - z$ we obtain

$$x^2 + z^2 = r^2 = (z + \eta)^2 = z^2 + 2\eta z + \eta^2$$

(20)

i.e.

$$z = \frac{x^2 - \eta^2}{2\eta}$$

(21)

which is the equation of a parabola pointing upward. With similar considerations we can
show that the lines $\xi = \text{constant}$ have equation
\[ z = \frac{\xi^2 - x^2}{2\xi} \] (22)
and are parabolae pointing downward. From Eqs. 21 and 22 we get
\[ dz = \frac{x \, dx}{\eta} \] (23)
\[ dz = -\frac{x \, dx}{\xi} \] (24)
The $x$ coordinate at the point of intersection satisfies
\[ \frac{\xi^2 - x^2}{2\xi} = \frac{x^2 - \eta^2}{2\eta} \] (25)
from which we get
\[ x^2 = \xi \eta \] (26)
From Eqs. 23 and 24 we see that the product of the two slopes at the point of intersection is
\[ -\frac{x^2}{\xi \eta} \] (27)
which on account of Eq. 26 is equal to $-1$. Thus the two curves intersect at an angle of $90^\circ$.

B) It is convenient to start from the expression of the Laplacian in cylindrical coordinates
\[ \Delta = \frac{1}{\rho} \frac{\partial}{\partial \rho} \rho \frac{\partial}{\partial \rho} + \frac{\partial^2}{\partial z^2} + \frac{1}{\rho^2} \frac{\partial^2}{\partial \phi^2} \] (28)
and change coordinates to
\[ \xi = \sqrt{\rho^2 + z^2 + z} \] (29)
\[ \eta = \sqrt{\rho^2 + z^2 - z} \] (30)
where $\rho = \sqrt{r^2 - z^2}$. We have
\[ \frac{\partial}{\partial \rho} = \frac{\partial}{\partial \xi} \frac{\partial}{\partial \xi} + \frac{\partial}{\partial \eta} \frac{\partial}{\partial \eta} + \frac{\partial}{\partial \xi} \frac{\partial}{\partial \xi} = \frac{\sqrt{r^2 - z^2}}{r} \left[ \frac{\partial}{\partial \xi} + \frac{\partial}{\partial \eta} \right] = \frac{2\sqrt{\xi \eta}}{\xi + \eta} \left[ \frac{\partial}{\partial \xi} + \frac{\partial}{\partial \eta} \right] \] (31)
\[ \frac{\partial}{\partial z} = \frac{\partial}{\partial \xi} \frac{\partial}{\partial \xi} + \frac{\partial}{\partial \eta} \frac{\partial}{\partial \eta} + \frac{\partial}{\partial \xi} \frac{\partial}{\partial \xi} = \frac{\sqrt{r^2 + z^2}}{r} \left[ \frac{\partial}{\partial \xi} + \frac{\partial}{\partial \eta} \right] = \frac{\xi - \eta}{\xi + \eta} \left[ \frac{\partial}{\partial \xi} + \frac{\partial}{\partial \eta} \right] + \frac{\partial}{\partial \xi} - \frac{\partial}{\partial \eta} = \frac{2\xi}{\xi + \eta} \frac{\partial}{\partial \xi} - \frac{2\eta}{\xi + \eta} \frac{\partial}{\partial \eta} \] (32)
Substituting into Eq. 28 and using $\rho = \sqrt{r^2 - z^2} = \sqrt{\xi \eta}$ we obtain

$$
\Delta = \frac{1}{\sqrt{\xi \eta}} \frac{2\sqrt{\xi \eta}}{\xi + \eta} \left( \frac{\partial}{\partial \xi} + \frac{\partial}{\partial \eta} \right) \sqrt{\xi \eta} \frac{2\sqrt{\xi \eta}}{\xi + \eta} \left( \frac{\partial}{\partial \xi} + \frac{\partial}{\partial \eta} \right)
$$

$$
+ \left( \frac{2\xi}{\xi + \eta} \frac{\partial}{\partial \xi} - \frac{2\eta}{\xi + \eta} \frac{\partial}{\partial \eta} \right) \left( \frac{2\xi}{\xi + \eta} \frac{\partial}{\partial \xi} - \frac{2\eta}{\xi + \eta} \frac{\partial}{\partial \eta} \right) + \frac{1}{\xi \eta} \frac{\partial^2}{\partial \phi^2} =
$$

$$
\frac{4}{\xi + \eta} \left[ \left( \frac{\partial}{\partial \xi} + \frac{\partial}{\partial \eta} \right) \frac{\xi \eta}{\xi + \eta} + \left( \frac{\xi}{\xi + \eta} \frac{\partial}{\partial \xi} - \frac{\eta}{\xi + \eta} \frac{\partial}{\partial \eta} \right) \frac{\xi}{\xi + \eta} \right] \frac{\partial}{\partial \xi}
$$

$$
+ \left[ \left( \frac{\partial}{\partial \xi} + \frac{\partial}{\partial \eta} \right) \frac{\xi \eta}{\xi + \eta} - \left( \frac{\xi}{\xi + \eta} \frac{\partial}{\partial \xi} - \frac{\eta}{\xi + \eta} \frac{\partial}{\partial \eta} \right) \frac{\eta}{\xi + \eta} \right] \frac{\partial}{\partial \eta}
$$

$$
+ \left[ \left( \frac{\partial}{\partial \xi} + \frac{\partial}{\partial \eta} \right) \frac{\xi \eta}{\xi + \eta} \right] \frac{\partial^2}{\partial \xi^2} + \frac{\partial^2}{\partial \eta^2} + \frac{\partial}{\partial \xi} + \frac{\partial}{\partial \eta} + \frac{1}{\xi \eta} \frac{\partial^2}{\partial \phi^2} =
$$

$$
\frac{4}{\xi + \eta} \left[ \frac{\partial^2}{\partial \xi^2} + \frac{\partial^2}{\partial \eta^2} + \frac{\partial}{\partial \xi} + \frac{\partial}{\partial \eta} \right] + \frac{1}{\xi \eta} \frac{\partial^2}{\partial \phi^2}
$$

(33)

**Problem 3:**

Consider the Schrödinger equation for the hydrogen atom, with units such that $\hbar = m = e = 1$

$$
-\frac{1}{2} \Delta \psi - \frac{1}{r} \psi = E \psi
$$

(34)

In parabolic coordinates this equation takes the form

$$
\frac{4}{\xi + \eta} \left[ \frac{\partial}{\partial \xi} \left( \xi \frac{\partial \psi}{\partial \xi} \right) + \frac{\partial}{\partial \eta} \left( \eta \frac{\partial \psi}{\partial \eta} \right) \right] + \frac{1}{\xi \eta} \frac{\partial^2 \psi}{\partial \phi^2} + 2 \left( E + \frac{2}{\xi + \eta} \right) \psi = 0
$$

(35)

Look for solutions of the form

$$
\psi(\xi, \eta, \phi) = f_1(\xi) f_2(\eta) e^{im\phi}
$$

(36)

and use the method of separation of variables to obtain two eigenvalue ODEs which should be satisfied by $f_1(\xi), f_2(\eta)$.

*Hint:* Make the substitution, multiply by $(\xi + \eta)/4$, and divide by $f_1(\xi) f_2(\eta)$. You will see that the whole expression separates into two functions which must be individually equal to a constant.
Solution

We substitute the expression for \( \phi \) given in Eq. 36 into the Schrödinger equation and multiply across by \( (\xi + \eta)/4 \). We obtain

\[
f_2(\eta) \frac{d}{d\xi} \left( \xi \frac{df_1(\xi)}{d\xi} \right) + f_1(\xi) \frac{d}{d\eta} \left( \eta \frac{df_2(\eta)}{d\eta} \right) - \frac{\xi + \eta}{4\xi\eta} m^2 f_1(\xi) f_2(\eta) + \frac{1}{2} \left( E(\xi + \eta) + 1 \right) f_1(\xi) f_2(\eta) = 0 
\]

or, dividing by \( f_1(\xi) f_2(\eta) \)

\[
\frac{1}{f_1(\xi)} \frac{d}{d\xi} \left( \xi \frac{df_1(\xi)}{d\xi} \right) + \frac{1}{f_2(\eta)} \frac{d}{d\eta} \left( \eta \frac{df_2(\eta)}{d\eta} \right) - \frac{m^2}{4\xi} - \frac{m^2}{4\eta} + \frac{E\xi}{2} + \frac{E\eta}{2} + \frac{1}{2} = 0 
\]

The dependence on the variables \( \xi \) and \( \eta \) in this equation can only be satisfied if

\[
\frac{1}{f_1(\xi)} \frac{d}{d\xi} \left( \xi \frac{df_1(\xi)}{d\xi} \right) - \frac{m^2}{4\xi} + \frac{E\xi}{2} + \beta_1 = 0 
\]

\[
\frac{1}{f_2(\eta)} \frac{d}{d\eta} \left( \eta \frac{df_2(\eta)}{d\eta} \right) - \frac{m^2}{4\eta} + \frac{E\eta}{2} + \beta_2 = 0 
\]

where \( \beta_1 \) and \( \beta_2 \) are two constants adding up to 1. Equations 39 and 40, with boundary conditions \( f_1(\xi) \sim \xi^{\left|m/2\right|} \) for \( \xi \to 0 \), \( f_1(\xi) \to 0 \) exponentially for \( \xi \to \infty \), and similarly for \( f_2(\eta) \), which can be deduced from the requirement that the solutions are normalizable, are two eigenvalue equations for the functions \( f_1 \) and \( f_2 \). One can show that their eigenfunctions can be labelled by two integers, \( n_1 \) and \( n_2 \) which are related to the principal quantum number \( n \) in the spectrum of the hydrogen atom by

\[
n = n_1 + n_2 + |m| + 1 
\]

Problem 4: After separation of variables in spherical coordinates, the radial Schrödinger equation for a particle of mass \( m \) moving in a central potential \( V(r) \) with angular momentum \( \ell \) is

\[
-\frac{1}{2m} \frac{d^2\psi(r)}{dr^2} - \frac{1}{mr} \frac{d\psi(r)}{dr} + \frac{\ell(\ell + 1)}{2mr^2} \psi(r) + V(r)\psi(r) = E\psi(r) 
\]

in units where \( \hbar = 1 \). Imagine that the potential \( V(r) \) is less or equal to zero with \( V(0) = -V_0 \) and of short range and that we look for possible negative energy bound states. In order to simplify the notation, let us rewrite Eq. 42 as

\[
\frac{d^2\psi(r)}{dr^2} + \frac{2}{r} \frac{d\psi(r)}{dr} - \left[ \frac{\ell(\ell + 1)}{r^2} + \nu(r) + \epsilon \right] \psi(r) = 0 
\]
with \( v(r) = 2mV(r), \epsilon = -2mE. \)

A) Use the Frobenius method to find solutions to Eq. 43 in the form of a power series

\[
\psi(r) = r^s + c_1 r^{s+1} + c_2 r^{s+2} + \ldots
\]  

(44)

assuming that \( v(r) \) has a power series expansion \( v(r) = -v_0 + v_1 r + \ldots. \)

Of the two solutions for \( s \) of the indicial equation, which one has an acceptable physical behavior for \( r \to 0? \) Find the first three terms of the corresponding power series expansion.

B) What is the behavior of the solutions to Eq. 43 for \( r \to \infty. \) Which one of the two possible behaviors for \( r \to \infty \) is acceptable on physical grounds?

C) Assume that you are able to integrate numerically with high accuracy the radial Schrödinger from \( r = 0 \) starting from the initial conditions you found in A), and to integrate it numerically down from very large \( r \) starting from the asymptotic behavior you determined in B). Denote the result of the two integrations by \( f_1(r, \epsilon) \) (for the integration up from \( r = 0 \)) and \( f_2(r, \epsilon) \) (for the integration down from very large \( r \)), where we added the dependence on \( \epsilon \) because you can carry out the numerical integrations for any value of \( \epsilon \). How would you go about to find the possible energy eigenvalues \( \epsilon \)?

Hint: Carry out the two integrations to a common, mid-range, \( r = r_0 \) and evaluate the Wronskian of the two solutions ...

Solution

A) From the power series expansion of Eq. 44 we find

\[
\psi'(r) = sr^{s-1} + c_1(s+1)r^s + c_2(s+2)r^{s+1} + \ldots
\]

(45)

\[
\psi''(r) = s(s-1)r^{s-2} + c_1(s+1)sr^{s-1} + c_2(s+2)(s+1)r^s + \ldots
\]

(46)

Substituting into Eq. 43 we find

\[
\begin{align*}
  s(s-1)r^{s-2} + c_1(s+1)sr^{s-1} + c_2(s+2)(s+1)r^s + \ldots \\
  +2[sr^{s-2} + c_1(s+1)r^{s-1} + c_2(s+2)r^s + \ldots] \\
  -\ell(\ell + 1)[r^{s-2} + c_1r^{s-1} + c_2r^s + \ldots] \\
  (-v_0 + \epsilon)r^s + \cdots = 0
\end{align*}
\]

(47)

By equating to zero the coefficient of the term \( r^{s-2} \) we obtain the indicial equation

\[
s(s-1) + 2s - \ell(\ell + 1) = s(s+1) - \ell(\ell + 1) = 0
\]

(48)

with solutions \( s_1 = \ell, \ s_2 = -\ell - 1. \) \( s_2 \) leads to a singular behavior of the wave function at the origin and must be rejected. Thus the radial wave function will have an expansion

\[
\psi(r) = r^{\ell} + c_1 r^{\ell+1} + c_2 r^{\ell+2} + \ldots
\]

(49)
Moreover, by looking at the terms with $r^{s-1}$ in Eq. 47 we conclude that $c_1 = 0$. By equation to zero the overall coefficient of $r^s$ we get

\[\left[(\ell + 2)(\ell + 1) + 2(\ell + 2) - \ell(\ell + 1)\right]c_2 - v_0 + \epsilon = [4\ell + 6]c_2 - v_0 + \epsilon = 0 \quad (50)\]

from which we obtain

\[c_2 = \frac{v_0 - \epsilon}{4\ell + 6} \quad (51)\]

The determination of the other coefficients in the expansion of $\psi(r)$ requires knowledge of $v(r)$.

B) The leading terms for $r \to \infty$ in Eq. 43 are the first term, with the second derivative of $\psi$, and the last term $\epsilon \psi$, because $v(r)$ is by assumption short ranged and the other two terms are suppressed by inverse powers of $r$. Thus for $r \to \infty$ the equation reduces to

\[\frac{d^2\psi(r)}{dr^2} + \epsilon \psi(r) = 0 \quad (52)\]

with solutions

\[\psi(r) \sim e^{\pm \sqrt{\epsilon}r} \quad (53)\]

(we defined $\epsilon$ to be positive for bound states.) Obviously only the exponentially decreasing behavior is acceptable, since the other one leads to a non-normalizable wave function.

C) An energy eigenfunction $\psi(r)$ will behave like $f_1(r)$ as well as $f_2(r)$, so if $\epsilon$ is an energy eigenvalue, the results of the two integrations must be proportional and the Wronskian of the two solutions $f_1(r, \epsilon)$ and $f_2(r, \epsilon)$ will vanish. So we can integrate the solutions up to a common $r = r_0$ and calculate their Wronskian $W$. $W$ will be a function of $\epsilon$ and we can find the energy eigenvalues, if any, by searching for zeros of $W(\epsilon)$. 