Problem 1:
A) Find the characteristics of the differential equation
\[ \frac{\partial^2 \phi(x, t)}{\partial t^2} - v(x)^2 \frac{\partial^2 \phi(x, t)}{\partial x^2} = 0 \] (1)
where
\[ v(x) = c \frac{x^2 + \ell^2}{x^2 + 2\ell^2} \] (2)
*Hint:* The characteristics are the trajectories of a particle moving with velocity \( \pm v(x) \).
B) Consider a perturbation at \( x = t = 0 \) which propagates toward positive \( x \) according to Eq. 1. How far behind would it lag for \( t \to \infty \) with respect to a ray of light also emitted at \( x = t = 0 \) and propagating in the positive \( x \) direction with constant velocity \( c \)?
C) Draw a space-time diagram of the trajectories of the perturbation and of the ray of light with \( c = 1 \) and \( \ell = 1 \). (Place the \( t \)-axis in vertically in the drawing.)

Problem 2:
This problem and problem two deal with the separation of variables in parabolic coordinates for a quantum system with Coulomb potential. They draw from “Quantum Mechanics-Nonrelativistic Theory” by L. D. Landau and E. M. Lifshitz.

In three-dimensional space introduce three curvilinear coordinates \( \xi, \eta, \phi \), called “parabolic coordinates” related to the Cartesian coordinates by
\[ x = \sqrt{\xi \eta} \cos \phi \] (3)
\[ y = \sqrt{\xi \eta} \sin \phi \] (4)
\[ z = \frac{1}{2} (\xi - \eta) \] (5)
\[ r = \sqrt{x^2 + y^2 + z^2} = \frac{1}{2} (\xi + \eta) \] (6)
with \( \leq \xi, \eta < \infty \); or, conversely,
\[ \xi = r + z \] (7)
\[ \eta = r - z \] (8)
\[ \phi = \phi = \arctan \frac{y}{x} \] (9)
A) The following is true for all $\phi = \text{constant}$ planes, but to be specific consider the $\phi = 0$ plane, i.e. the $xz$-plane with $x \geq 0$. Show that the coordinate lines $\xi = \text{constant}$ and $\eta = \text{constant}$ are parabolae (more precisely one of the two branches of a parabola) which meet at right angles. Plot or draw the coordinate lines with $\xi$ and $\eta$ equal to 5, 10, 15, 20.

B) Calculate the expression of the Laplacian in parabolic coordinates. As a check you should find that it is given by

$$\Delta = \frac{4}{\xi + \eta} \left[ \frac{\partial}{\partial \xi} \left( \xi \frac{\partial}{\partial \xi} \right) + \frac{\partial}{\partial \eta} \left( \eta \frac{\partial}{\partial \eta} \right) \right] + \frac{1}{\xi \eta} \frac{\partial^2}{\partial \phi^2} \quad (10)$$

**Hint:** I found it convenient to start from the expression of the Laplacian in cylindrical coordinates

$$\Delta = \frac{1}{\rho} \frac{\partial}{\partial \rho} \rho \frac{\partial}{\partial \rho} + \frac{\partial^2}{\partial z^2} + \frac{1}{\rho^2} \frac{\partial^2}{\partial \phi^2} \quad (11)$$

changing then coordinates to

$$\xi = \sqrt{\rho^2 + z^2} + z \quad (12)$$

$$\eta = \sqrt{\rho^2 + z^2} - z \quad (13)$$

where $\rho = \sqrt{r^2 - z^2}$. The algebra is demanding and must be done with care, but it is manageable.

**Problem 3:**

Consider the Schrödinger equation for the hydrogen atom, with units such that $\hbar = m = e = 1$

$$-\frac{1}{2} \Delta \psi - \frac{1}{r} \psi = E \psi \quad (14)$$

In parabolic coordinates this equation takes the form

$$\frac{4}{\xi + \eta} \left[ \frac{\partial}{\partial \xi} \left( \xi \frac{\partial \psi}{\partial \xi} \right) + \frac{\partial}{\partial \eta} \left( \eta \frac{\partial \psi}{\partial \eta} \right) \right] + \frac{1}{\xi \eta} \frac{\partial^2 \psi}{\partial \phi^2} + 2 \left( E + \frac{2}{\xi + \eta} \right) \psi = 0 \quad (15)$$

Look for solutions of the form

$$\psi(\xi, \eta, \phi) = f_1(\xi) f_2(\eta) e^{im\phi} \quad (16)$$

and use the method of separation of variables to obtain two eigenvalue ODEs which should be satisfied by $f_1(\xi)$, $f_2(\eta)$.

**Hint:** Make the substitution, multiply by $(\xi + \eta)/4$, and divide by $f_1(\xi) f_2(\eta)$. You will see that the whole expression separates into two functions which must be individually equal to a constant.

Please note: Your problem ends here, but when I post the solutions read the last paragraph in the solution which relates the two integer quantum numbers emerging from the above eigenvalue ODEs to the principal quantum number in the spectrum the hydrogen atom.
Problem 4: After separation of variables in spherical coordinates, the radial Schrödinger equation for a particle of mass $m$ moving in a central potential $V(r)$ with angular momentum $\ell$ is

$$-\frac{1}{2m} \frac{d^2 \psi(r)}{dr^2} - \frac{1}{mr} \frac{d \psi(r)}{dr} + \frac{\ell(\ell + 1)}{2mr^2} \psi(r) + V(r) \psi(r) = E \psi(r)$$  \hspace{1cm} (17)$$

in units where $\hbar = 1$. Imagine that the potential $V(r)$ is less or equal to zero with $V(0) = -V_0$ and of short range and that we look for possible negative energy bound states. In order to simplify the notation, let us rewrite Eq. 17 as

$$\frac{d^2 \psi(r)}{dr^2} + \frac{2}{r} \frac{d \psi(r)}{dr} - \left[ \frac{\ell(\ell + 1)}{r^2} + v(r) + \epsilon \right] \psi(r) = 0$$  \hspace{1cm} (18)$$

with $v(r) = 2mV(r)$, $\epsilon = -2mE$.

A) Use the Frobenius method to find solutions to Eq. 18 in the form of a power series

$$\psi(r) = r^s + c_1 r^{s+1} + c_2 r^{s+2} + \ldots$$  \hspace{1cm} (19)$$

assuming that $v(r)$ has a power series expansion $v(r) = -v_0 + v_1 r + \ldots$.

Of the two solutions for $s$ of the indicial equation, which one has an acceptable physical behavior for $r \to 0$? Find the first three terms of the corresponding power series expansion.

B) What is the behavior of the solutions to Eq. 18 for $r \to \infty$. Which one of the two possible behaviors for $r \to \infty$ is acceptable on physical grounds?

C) Assume that you are able to integrate numerically with high accuracy the radial Schrödinger from $r = 0$ starting from the initial conditions you found in A), and to integrate it numerically down from very large $r$ starting from the asymptotic behavior you determined in B). Denote the result of the two integrations by $f_1(r, \epsilon)$ (for the integration up from $r = 0$) and $f_2(r, \epsilon)$ (for the integration down from very large $r$), where we added the dependence on $\epsilon$ because you can carry out the numerical integrations for any value of $\epsilon$. How would you go about to find the possible energy eigenvalues $\epsilon$?

Hint: Carry out the two integrations to a common, mid-range, $r = r_0$ and evaluate the Wronskian of the two solutions ...