Do all four problems. Each correct solution will be given 25 points, with points taken away for errors (or for absence of explanation) according to the severity of the error.

**Problem 1:**

The equation

\[ w = \frac{1}{z} \]  \hspace{1cm} (1)

defines a conformal mapping from the complex \( z \)-plane to the complex \( w \)-plane in any finite region in the \( z \)-plane which excludes the origin.

A) Prove that straight lines in the \( z \)-plane which do not pass through the origin are mapped into circles in the \( w \)-plane which pass through the origin.

B) The real axis in the \( z \)-plane is obviously mapped into the real axis in the \( w \)-plane. Consider the line defined by

\[ z = 2 + te^{i\phi} \]  \hspace{1cm} (2)

with \(-\infty < t < \infty \) and fixed \( \phi \neq 0 \), which forms an angle \( \phi \) with the real axis, and show that it is mapped into a circle which intersects the positive real axis in the \( w \)-plane at an angle \( \phi \).

**Problem 2:**

Consider the forced harmonic oscillator equation

\[ \frac{d^2y(t)}{dt^2} + \omega_0^2 y(t) = F(t) \]  \hspace{1cm} (3)

and assume that the system is at rest with \( y(t) = 0, dy(t)/dt = 0 \) and \( F(t) = 0 \) for \(-\infty < t \leq 0\).

Use the method of variation of the constants to find the solution of the equation with the following driving force:

\[ F(t) = \sin \omega t \]  \hspace{1cm} for \( t \geq 0 \)  \hspace{1cm} (4)

In particular, show that if \( \omega = \omega_0 \) the amplitude of the oscillation grows without bound.

Important note: you must find the explicit form of the solution; you may not leave it in an implicit integral representation.

*Hint:* The solution of this problem is, in principle, straightforward, but the algebra of the trigonometric functions can become quite involved. I found it easier to solve the problem...
with a driving force \( \exp(\omega t) \), taking at the end the imaginary part of the solution. Also I found convenient to use \( \exp(\omega_0 t) \) and \( \exp(-\omega_0 t) \) as the two solutions of the homogeneous equation.

**Optional:** Do a few plots of the solution for \( \omega \) smaller, equal and larger than \( \omega_0 \). This is for your own interest. Do not include the graphs into the pdf file for the assignment, but if you would like to show them to me, send them as attachments to a separate message and I will be very pleased to look at them.

**Problem 3:**
The equation

\[
x^2 \frac{d^2 y(x)}{dx^2} - x \frac{dy(x)}{dx} - 3y(x) = 0
\]

has solutions

\[
y_1(x) = x^{s_1} \\
y_2(x) = x^{s_2}
\]

A) Find the two different values \( s_1, s_2 \) which will make the equation satisfied.
B) Calculate the Wronskian \( W(x) \) of the two solutions using the definition of \( W \) in Equation 7 of the lecture notes on “Second order, linear, ordinary differential equations.”
C) Calculate \( W(x) \) according to Equation 14 of the same set of lecture notes and check that the two results agree.

**Problem 4:**
The equation

\[
x^2 y'' + 2xy' + (x^2 - 2)y = 0
\]

has a solution

\[
y_1(x) = \frac{\sin x}{x^2} - \frac{\cos x}{x}
\]

(this is known as the “spherical Bessel function of order one.”)

A) Verify that \( y_1(x) \) solves the equation.
B) Calculate the Wronskian, up to an arbitrary constant. (Remember to cast the equation in the form \( y'' + P(x)y' + Q(x)y = 0 \).)
C) Prove the following identity:

\[
\frac{d}{dx} \left( \frac{-\cos x - x \sin x}{\sin x - x \cos x} \right) = \frac{x^2}{(\sin x - x \cos x)^2}
\]

D) Use the method of “variation of the constant” to find the solution

\[
y_2(x) = -\frac{\cos x}{x^2} - \frac{\sin x}{x}
\]
(this is known as the “spherical Neumann function of order one.”)

*Hint:* the method of “variation of the constant” leads to an integral which can be explicitly calculated by using the identity in part C.