**PY 501 - Mathematical Physics**

Assignment 3 - September 24, 2020.
To be completed by the end of the day on Thursday, October 1, 2020.

Do all four problems. Each correct solution will be given 25 points, with points taken away for errors (or for absence of explanation) according to the severity of the error.

**Problem 1:**
Calculate the integral

\[ I = \int_{0}^{\infty} \frac{1}{\sqrt{x} (x^2 + a^2)} \, dx \]  
(1)
in two different ways.

a) By expressing the integral in terms of a contour integral of the function

\[ f(z) = \frac{1}{\sqrt{z} (z^2 + a^2)} \]  
(2)
in the complex \( z \) plane with a cut along the negative \( z \) axis.

b) By performing the change of variables \( x = y^2 \) which gives

\[ I = \int_{0}^{\infty} \frac{2}{y^4 + a^2} \, dy = \int_{-\infty}^{\infty} \frac{1}{y^4 + a^2} \, dy \]  
(3)
and evaluating the integral by the residue theorem after closing the path of integration with a semicircle going to \( +i\infty \) (or \( -i\infty \), which gives the same result.)

**Solution**

a) Let us work in the Riemann sheet where the square root of \( z \) is positive on the real axis. It is convenient to express \( z \) in terms of its magnitude and phase:

\[ z = re^{i\phi} \]  
(4)
Consider a contour \( \gamma \) which comes from \( z = -\infty \) just above the cut along the negative real axis, goes around the origin on the side of the positive real axis, goes back to \( z = -\infty \) just below the cut along the negative real axis, and then describes a full circle at \( r = \infty \) returning to the original point of departure. (We use freely the word “infinity” to represent a very large number \( R \) which will be eventually sent to infinity.) We have

\[ \oint f(z) \, dz = \int_{-\infty}^{0} \frac{1}{\sqrt{z} (z^2 + a^2)} \, dz + \int_{0}^{\infty} \frac{1}{\sqrt{z} (z^2 + a^2)} \, dz \]  
(5)
since the contribution from the circle at infinity vanishes. In both integrals \( z = -r \), so we can write

\[
\oint f(z) \, dz = \int_{0}^{\infty} \frac{1}{\sqrt{z} (z^2 + a^2)} \, dr - \int_{0}^{\infty} \frac{1}{\sqrt{z} (z^2 + a^2)} \, dr
\]

where \( z = r \exp(i\pi) \) in the first integral and \( z = r \exp(-i\pi) \) in the second one. Consequently \( \sqrt{z} \) will have value \( z = r \exp(\frac{i\pi}{2}) = i \) in the first integral and \( z = r \exp(-\frac{i\pi}{2}) = -i \) in the second one, giving

\[
\oint f(z) \, dz = -i \int_{0}^{\infty} \frac{1}{\sqrt{r} (r^2 + a^2)} \, dr - i \int_{0}^{\infty} \frac{1}{\sqrt{r} (r^2 + a^2)} \, dr = 2iI
\]

On the other hand the contour \( \gamma \) encloses the two poles at \( z = ia \) and \( z = -ia \) and so the contour integral can be done with the theorem of residues. We get

\[
\oint f(z) \, dz = 2\pi i \left[ \text{Res} \left( \frac{1}{\sqrt{z} (z - ia)(z + ia)}, z = ia \right) + \text{Res} \left( \frac{1}{\sqrt{z} (z - ia)(z + ia)}, z = -ia \right) \right]
\]

In this expression we have

\[
1/\sqrt{z} = z^{-1/2} = (ae^{i\pi/2})^{-1/2} = \frac{e^{-i\pi/4}}{\sqrt{a}} = \frac{\cos(\pi/4) - i \sin(\pi/4)}{\sqrt{a}} = \frac{1 - i}{\sqrt{2a}}
\]

in the first residue, and

\[
1/\sqrt{z} = z^{-1/2} = (ae^{-i\pi/2})^{-1/2} = \frac{e^{i\pi/4}}{\sqrt{a}} = \frac{\cos(\pi/4) + i \sin(\pi/4)}{\sqrt{a}} = \frac{1 + i}{\sqrt{2a}}
\]

in the second residue. Consequently we find

\[
\oint f(z) \, dz = 2\pi i \left( \frac{1 - i}{\sqrt{2a} (2ia)} + \frac{1 + i}{\sqrt{2a} (-2ia)} \right) = \frac{\sqrt{2} \pi i}{a^{3/2}}
\]

Comparing Eqs. 7 and 11 we finally find

\[
2iI = \frac{\sqrt{2} \pi i}{a^{3/2}}
\]

or

\[
I = \frac{\pi}{\sqrt{2a^3}}
\]

b) We close the integral with a semicircle at \( +i\infty \) converting \( I \) to a contour integral over a closed path \( \gamma \)

\[
I_{\gamma} = \oint f(y) \, dy
\]
with

\[ f(y) = \frac{1}{y^4 + a^2} \]  

(15)

The polynomial \( y^4 + a^2 \) has four zeroes at

\[ y_1 = \sqrt{a} e^{i\pi/4} = \sqrt{a} \frac{1+i}{\sqrt{2}} \]  

(16)

\[ y_2 = \sqrt{a} e^{3i\pi/4} = \sqrt{a} \frac{-1+i}{\sqrt{2}} \]  

(17)

\[ y_3 = \sqrt{a} e^{5i\pi/4} = \sqrt{a} \frac{-1-i}{\sqrt{2}} \]  

(18)

\[ y_4 = \sqrt{a} e^{7i\pi/4} = \sqrt{a} \frac{1-i}{\sqrt{2}} \]  

(19)

Correspondingly \( f(y) \) has simple poles at \( y_1, y_2, y_3, y_4 \). The path \( \gamma \) encloses the poles at \( y_1 \) and \( y_2 \). The theorem of residues gives then

\[
I_\gamma = \oint f(y) dy = 2\pi i \left( \text{Res} \ (f(y), y = y_1) + \text{Res} \ (f(y), y = y_2) \right) =
\]

\[
2\pi i \left( \frac{1}{(y_1 - y_2)(y_1 - y_3)(y_1 - y_4)} + \frac{1}{(y_2 - y_1)(y_2 - y_3)(y_2 - y_4)} \right) =
\]

\[
2\pi i \frac{2\sqrt{2}}{a^3} \left( \frac{1}{2(2 + 2i)(-2 + 2i)} + \frac{1}{(-2)(2i)(-2 + 2i)} \right) =
\]

\[
2\pi i \frac{2\sqrt{2}}{a^3} \left( -\frac{1}{8i} + \frac{1}{-8 + 8i} \right) = \frac{\pi}{\sqrt{2}a^3}
\]  

(20)

**Problem 2:**

Prove the identity

\[
\int_{-\infty}^{\infty} \frac{e^{\alpha x}}{1 + e^x} \, dx = \frac{\pi}{\sin \pi a} \quad (0 < |a| < 1)
\]  

(21)

by expressing the integral in terms of a contour integral over the rectangle with sides along the lines \( y = 0, y = 2\pi, x = -R, \) and \( x = R, \) sending then \( R \to \infty. \)

**Solution**

Consider the function

\[ f(z) = \frac{e^{az}}{1 + e^z} \]  

(22)

and the contour integral

\[ I = \oint f(z) \, dz \]  

(23)
where $\gamma$ traces the sides of the rectangle in counterclockwise direction starting from $-R, 0$. When $R$ is sent to $\infty$ the contribution to $I$ from the vertical sides of the rectangle will vanish and we are left with

$$I = \int_{-\infty}^{\infty} \frac{e^{ax}}{1 + e^x} \,dx + \int_{-\infty}^{-\infty} \frac{e^{a(x+2\pi i)}}{1 + e^{x+2\pi i}} \,dx = (1 - e^{2\pi i a}) \int_{-\infty}^{\infty} \frac{e^{ax}}{1 + e^x} \,dx$$ \hspace{1cm} (24)

The contour encloses the pole of $f(z)$ for $z = \pi i$ ($\exp(\pi i) = -1$) with residue $-\exp(\pi i a)$. From the residue theorem we get

$$I = 2\pi i \text{Res} \left( f(z), z = \pi i \right) = -2\pi i e^{\pi i a}$$ \hspace{1cm} (25)

Comparing Eqs. 24 and 25 we find

$$\int_{-\infty}^{\infty} \frac{e^{ax}}{1 + e^x} \,dx = \frac{-2\pi i e^{\pi i a}}{1 - e^{2\pi i a}} = \frac{\pi}{\sin \pi a}$$ \hspace{1cm} (26)

**Problem 3:**

Show that the function

$$v(x, y) = \frac{\sin 2x}{\cosh 2y + \cos 2x}$$ \hspace{1cm} (27)

can serve as the imaginary part of an analytic function $f(z)$ and find $f(z)$.

*Hint* It should be possible to show that $v(x, y)$ is a harmonic function by calculating $\Delta v$ and then find the conjugate harmonic function $u$, but when I solved the problem I did not have much taste for calculating the first and second derivatives of $v$. I tried instead to follow the notion that an analytic function is a function of a complex variable only and not of a variable and its complex conjugate. You do not have to solve the problem the way I did it, of course, but I illustrate my way of thinking in the hope that it may be useful.

Let us express $v$ in terms of exponentials

$$v(x, y) = -i \frac{e^{2ix} - e^{-2ix}}{e^{2y} + e^{-2y} + e^{2ix} + e^{-2ix}}$$ \hspace{1cm} (28)

This suggests a change of variable

$$w = e^{-iz} = e^{-ix+y}$$ \hspace{1cm} (29)

With this we find

$$w^* = e^{iz} = e^{ix+y}, \quad e^{2ix} = \frac{w^*}{w}, \quad e^{-2ix} = \frac{w}{w^*}, \quad e^{2y} = ww^*$$ \hspace{1cm} (30)

and $v$ takes the form

$$v(x, y) = -i \frac{\frac{w^*}{w} - \frac{w}{w^*}}{ww^* + \frac{1}{ww^*} + \frac{w^*}{w} + \frac{w}{w^*}}$$ \hspace{1cm} (31)
or, multiplying numerator and denominator by $ww^*$

$$v(x, y) = -i \frac{w^* - w}{w^2 w^* + 1 + w^2 + w^*}$$

(32)

The question is then: is this expression the imaginary part of a function of $w$ only?

If you followed my route all the way to here, you should be able to proceed to the end.

**Solution**

We continue from Eq. 32 in the text of the assignment. We can further simplify the r.h.s. as

$$v(x, y) = -i \frac{w^* - w}{1 + w^2 w^* + w^2 + w^*} = -i \frac{1 + w^* - w}{(1 + w^2)(1 + w^2)} =$$

$$-i \left( \frac{1}{1 + w^2} - \frac{1}{1 + w^*} \right) = \text{Im} \frac{2}{1 + w^2}$$

(33)

This shows that $v(x, y)$ is the imaginary part of the analytic function

$$f(z) = \frac{2}{1 + e^{-2iz}}$$

(34)

Three considerations.

1) A posteriori it would have been possible to arrive to the same result by multiplying numerator and denominator in Eq. 28 directly by $\exp(2y)$, without resorting to the change of variable to $w$. This gives

$$v(x, y) = -i \frac{e^{2(y+ix)} - e^{2(y-ix)}}{e^{4y} + 1 + e^{2(y+ix)} + e^{2(y-ix)}}$$

(35)

from which one can follow the same steps outlined in the above solution without resorting to the change of variable. This occurred to me only after I solved the problem by going from $z$ to $w$.

2) It is easy to verify that the imaginary part of $f(z)$ is $v(x, y)$. We have

$$f(z) = \frac{2}{1 + e^{-2ix}} = \frac{2}{1 + e^{-2ix+2y}} = \frac{2}{1 + e^{2y}(\cos 2x - i \sin 2x)} =$$

$$\frac{2[1 + e^{2y}(\cos 2x + i \sin 2x)]}{(1 + e^{2y}(\cos 2x - i \sin 2x))(1 + e^{2y}(\cos 2x + i \sin 2x))} =$$

$$\frac{2[1 + e^{2y}(\cos 2x + i \sin 2x)]}{1 + e^{4y} + 2e^{2y} \cos 2x}$$

(36)

The imaginary part of $f(z)$ is therefore

$$\text{Im} \ f(z) = \frac{2e^{2y} \sin 2x}{1 + e^{4y} + 2e^{2y} \cos 2x} = \frac{2 \sin 2x}{e^{2y} + e^{-2y} + 2 \cos 2x} = \frac{\sin 2x}{\cosh 2y + \cos 2x}$$

(37)
as in Eq. 27.

3) The solution in Eq. 34 is equivalent to

\[ f(z) = i \tan z \]  

Indeed

\[ i \tan z = \frac{e^{iz} - e^{-iz}}{e^{iz} + e^{-iz}} = \frac{1 - e^{-2iz}}{1 + e^{-2iz}} = \frac{2 - (1 + e^{-2iz})}{1 + e^{-2iz}} = \frac{2}{1 + e^{-2iz}} - 1 \]  

which differs from the solution in Eq. 34 by a constant in the real part of \( f(z) \).

**Problem 4:**

The mapping

\[ z \rightarrow w = z^2 \]  

is not conformal at \( z = 0 \) because the inverse mapping \( z = \sqrt{w} \) is singular there. Indeed, with \( z = x + iy \) and \( w = u + iv \), the lines \( y = 0 \) and \( x = 0 \), which meet at a 90\(^\circ\) angle at \( z = 0 \) in the \( z \)-plane, are mapped to the lines \( v = 0, u \geq 0 \) and \( v = 0, u \leq 0 \) which form an angle of 180\(^\circ\) in the \( w \)-plane. However, in a region which excludes the origin and where the mapping is one to one, the mapping is conformal.

For this problem consider the wedge shaped region in the \( z \) plane delimited by the lines \( \alpha : y = 0, x > 0 \) (i.e positive real axis) and \( \beta : x = y, x > 0 \), which form an angle of 45\(^\circ\). Their images are the lines \( \gamma : v = 0, u > 0 \) and \( \delta : u = 0, v > 0 \) in the complex \( w \)-plane (i.e. the positive real and imaginary axes.) The lines

\[ x = a \]  

with \( a > 0 \) is the \( z \)-plane are lines parallel to the imaginary axis which meet the line \( \alpha \) at a 90\(^\circ\) degree angle and the line \( \beta \) at a 45\(^\circ\) angle. Find the images of these lines in the \( w \)-plane, expressed as an equation in \( u \) and \( v \). You should find that they are parabolas. Show these parabolas meet the line \( \gamma \) at a 90\(^\circ\) degree angle and the line \( \delta \) at a 45\(^\circ\) angle.

**Solution**

From the equation

\[ w = u + iv = z^2 = (x + iy)^2 = x^2 - y^2 + 2ixy \]  

we get

\[ u = x^2 - y^2 \]  

\[ v = 2xy \]  

If we take \( x = a \) and allow \( y \) to vary, Eqs. 43, 44 give a parametric representation of the parabola

\[ u = a^2 - \frac{v^2}{4a^2} \]  

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The parabola is symmetric with respect to the $u$-axis, so it meets the line $\gamma$, i.e. the $u$-axis, at an angle of $90^\circ$. The parabola meets the line $\delta$, i.e. the $v$-axis, at

$$v = 2a^2$$  \hspace{1cm} (46)

Equation 45 gives us

$$du = \frac{v}{2a^2} \, dv$$  \hspace{1cm} (47)

and so the parabola meets $\delta$ with slope

$$\frac{dv}{du} = \frac{2a^2}{v} = 1$$  \hspace{1cm} (48)

i.e. at $45^\circ$. 
