Do all four problems, however see the note below about problems 3 and 4. Each correct solution will be given 25 points, with points taken away for errors (or for absence of explanation) according to the severity of the error.

Problems 3 and 4 ask that one calculates some integrals, 8 for problem 3 and 6 for problem 4. Calculating them all is time consuming, so you only have to return the evaluation of 4 integrals of your own choosing for problem 3, and of 3 integrals, again of your own choosing, for problem 4. You may want to calculate also the other integrals or some of them as an exercise, and you will be able to check your work when I post the solutions, but only submit the requested numbers of evaluations for problems 3 and 4.

**Problem 1:**

The finite sum

\[ f(x) = \sum_{n=0}^{N} e^{inx} \]  \hspace{1cm} (1)

is a geometric progression and can easily be summed

\[ f(x) = \sum_{n=0}^{N} (e^{ix})^n = \frac{1 - e^{i(N+1)x}}{1 - e^{ix}} \]  \hspace{1cm} (2)

Use this result and Euler’s formula to prove the identity

\[ f(x) = \sum_{n=0}^{N} \cos nx = \frac{\sin \frac{1}{2}(N + 1)x}{\sin \frac{1}{2}x} \cos \frac{N}{2}x \] \hspace{1cm} (x = \text{real})  \hspace{1cm} (3)

What is the corresponding sum involving \( \sin nx \)?
Solution

We write the sum in the l.h.s. of Eq. 2 in the form

\[ \sum_{n=0}^{N} (e^{ix})^n = \sum_{n=0}^{N} (e^{inx}) = \sum_{n=0}^{N} (\cos nx + i \sin nx) \]  

(4)

On the other hand, with a little algebra we can rewrite the r.h.s. of the same equation in the form

\[ \frac{1 - e^{i(N+1)x}}{1 - e^{ix}} = \frac{e^{-ix/2} - e^{i(N+1)x/2}}{e^{-ix/2} - e^{ix/2}} e^{iN/2}e^{ix/2} \]

\[ = \frac{e^{-ix/2} - e^{i(N+1)x/2}}{e^{-ix/2} - e^{ix/2}} e^{iN/2} = \frac{\sin \frac{1}{2}(N + 1)x}{\sin \frac{1}{2}x} \left( \cos \frac{N}{2}x + i \sin \frac{N}{2}x \right) \]  

(5)

By equating real and imaginary parts in Eqs. 4 and 5 we find

\[ \sum_{n=0}^{N} \cos nx = \frac{\sin \frac{1}{2}(N + 1)x}{\sin \frac{1}{2}x} \cos \frac{N}{2}x \]  

(6)

\[ \sum_{n=0}^{N} \sin nx = \frac{\sin \frac{1}{2}(N + 1)x}{\sin \frac{1}{2}x} \sin \frac{N}{2}x \]  

(7)

Problem 2:

Consider the two functions

\[ u(x, y) = \frac{x(1 + x) + y^2}{(1 + x)^2 + y^2} \]  

(8)

\[ v(x, y) = \frac{y}{(1 + x)^2 + y^2} \]  

(9)

Can \( u(x, y) \) and \( v(x, y) \) be the real and imaginary part of an analytic function \( f(z) \)?

Answer either by checking whether \( u \) and \( v \) satisfy the Cauchy-Riemann conditions or by any other method of your choice.

If the answer is yes, what is \( f(z) \)?
Solution

Let us rewrite \( u(x, y) \) as

\[
\begin{align*}
\frac{x(1 + x) + y^2}{(1 + x)^2 + y^2} &= x(1 + x) + y^2 - (1 + x) + \frac{1}{(1 + x)^2 + y^2} = \\
&= \frac{x(1 + x) + y^2}{(1 + x)^2 + y^2} - \frac{1}{(1 + x)^2 + y^2} = \\
&= 1 - \frac{1}{(1 + x)^2 + y^2}.
\end{align*}
\]

Let us now consider the combined expression

\[
\begin{align*}
\frac{1 + x}{(1 + x)^2 + y^2} - \frac{1}{(1 + x)^2 + y^2} &= 1 - \frac{1 + x}{(1 + x)^2 + y^2} = \\
&= 1 - \frac{1}{1 + z},
\end{align*}
\]

This shows that \( u + iv \) is a function of \( z \) only (not of \( z \) and \( z^* \)) and is therefore an analytic function of \( z \) (with a pole at \( z = -1 \)). This answers also the last question in the problem: \( f(z) = 1 - 1/(z + 1) \).

Problem 3:

Using the Cauchy theorem, Cauchy Integral formula, or their consequences, evaluate the following integrals, all taken around the circle \(|z| = 2\):

\[
\begin{align*}
a) & \oint \frac{\cos z}{z} \, dz \\
b) & \oint \frac{\sin z}{z} \, dz \\
c) & \oint \frac{e^z}{z-1} \, dz \\
d) & \oint \frac{2z^2 + 3z - 1}{z^2 - 9} \, dz \\
e) & \oint \frac{z^2}{z^2 - 9} \, dz \\
f) & \oint \frac{2z}{z^2 - 9} \, dz \\
g) & \oint \frac{e^z}{(z-1)^2} \, dz \\
h) & \oint \frac{\sin z}{z^n} \, dz
\end{align*}
\]

Solution

Alle the following statements apply to the region bounded by the circle.

a) The integrand has a simple pole at \( z = 0 \) with residue \( \cos 0 = 1 \):

\[
\oint \frac{\cos z}{z} \, dz = 2\pi i
\]
b) The integrand is regular thus

\[ \oint \frac{\sin z}{z} \, dz = 0 \]  \hfill (13)

c) The integrand has a simple pole at \( z = 1 \) with residue \( e^1 = e \):

\[ \oint \frac{e^z}{z - 1} \, dz = 2\pi i e \]  \hfill (14)

d) The integrand has a simple pole at \( z = 1 - i \) with residue \( 2(1 - i)^2 + 3(1 - i) - 1 = 2 - 7i \)

\[ \oint \frac{2z^2 + 3z - 1}{z - 1 + i} \, dz = 2\pi i(2 - 7i) \]  \hfill (15)

e) The integrand has poles at \( z = \pm 3 \) but is regular inside the circle, therefore the integral vanishes.

f) The integrand has a simple pole at the origin with residue 1 \( (\sin z/z^2 = (z - z^3/6 + \ldots)/z^2 = 1/z + \ldots) \).

\[ \oint \frac{\sin z}{z^2} \, dz = 2\pi i \]  \hfill (16)

g) The integrand has a double pole at \( z = 1 \). To find the residue we expand \( e^z = e^{z-1} e = (1 + (z - 1) + \ldots) e \)

With this we get

\[ \oint \frac{e^z}{(z - 1)^2} \, dz = e \oint \left( \frac{1}{(z - 1)^2} + \frac{1}{z - 1} + \text{regular terms} \right) \, dz = 2\pi i e \]  \hfill (18)

h) The \( z^{-1} \) term in the expansion of the integrand

\[ \frac{\sin z}{z^n} = \frac{z - z^3/3! + z^5/5! + \ldots}{z^n} \]  \hfill (19)

vanishes in \( n \) is odd or \( n \leq 0 \). For even \( n \geq 2 \) that coefficient of the \( z^{-1} \) term is \( (-1)^{n/2+1}/(n - 1)! \) and the value of the integral is \( 2\pi i(-1)^{n/2+1}/(n - 1)! \).
Problem 4:

Establish the following results:

a) \( \int_{-\infty}^{\infty} \frac{dx}{x^4 + a^4} = \frac{\pi}{a^3 \sqrt{2}} \)

b) \( \int_{0}^{\infty} \frac{x^2}{x^6 + 1} \, dx = \frac{\pi}{6} \)

c) \( \int_{-\infty}^{\infty} \frac{dx}{(x^2 + 1)^3} = \frac{3\pi}{8} \)

d) \( \int_{-\infty}^{\infty} \frac{\cos kx \, dx}{(x-a)^2 + b^2} = \frac{\pi e^{-kb}}{b^e} \cos ka \quad (k > 0, b > 0) \)

e) \( \int_{0}^{\infty} \frac{\sin x \, dx}{x^2 + 1} = \frac{\pi}{2e} \)

f) \( \int_{-\infty}^{\infty} \frac{\cos x \, dx}{(x^2 + a^2)(x^2 + b^2)} = \frac{\pi}{a^2 - b^2} \left( e^{-b} - e^{-a} \right) \quad (a \neq b) \)

What is the result of f) if \( a = b? \)

Solution

a) \( I = \int_{-\infty}^{\infty} \frac{dx}{x^4 + a^4} = \int_{\gamma} \frac{1}{z^4 + a^4} \, dz \) \hspace{2cm} (20)

where \( \gamma \) consists of the real axis and a semicircle at infinity in the upper complex half plane. We have

\[ \frac{1}{z^4 + a^4} = \frac{1}{(z - z_1)(z - z_2)(z - z_3)(z - z_4)} \] \hspace{2cm} (21)

with

\[ z_1 = a \frac{1+i}{\sqrt{2}}, \quad z_2 = a \frac{-1+i}{\sqrt{2}}, \quad z_3 = a \frac{-1-i}{\sqrt{2}}, \quad z_4 = a \frac{1-i}{\sqrt{2}} \] \hspace{2cm} (22)

\( \gamma \) encloses the poles at \( z_1 \) and \( z_2 \). The theorem of residues gives

\[ I = 2\pi i \left( \frac{1}{z_1 - z_2} \right) \left( \frac{1}{z_1 - z_3} \right) \left( \frac{1}{z_1 - z_4} \right) + \frac{1}{(z_2 - z_1)(z_2 - z_3)(z_2 - z_4)} \]

\[ = \frac{2^{5/2}\pi i}{a^3} \left( \frac{1}{(2+2i)(2i)} + \frac{1}{(-2)(2i)(-2+2i)} \right) \]

\[ = \frac{2^{-1/2}\pi i}{a^3} \left( \frac{1}{-1+i} + \frac{1}{-1-i} \right) = \frac{\pi i}{a^3 \sqrt{2}} = \frac{\pi}{a^3 \sqrt{2}} \] \hspace{2cm} (23)

b) \( I = \int_{0}^{\infty} \frac{x^2}{x^6 + 1} \, dx = \frac{1}{2} \int_{-\infty}^{\infty} \frac{x^2}{x^6 + 1} \, dx \) \hspace{2cm} (24)

With \( y = x^3, \, dy = 3x^2 \, dx \)

\[ I = \frac{1}{6} \int_{-\infty}^{\infty} \frac{1}{y^2 + 1} \, dy \] \hspace{2cm} (25)
Close with a semicircle in the upper half plane and pick up the residue at $y = \frac{1}{2}$

\[ I = \frac{1}{6} \left( 2\pi i \frac{1}{2i} \right) = \frac{\pi}{6} \]  

(26)

c)  

\[ I = \int_{-\infty}^{\infty} \frac{dx}{(x^2 + 1)^3} = \oint_{\gamma} \frac{1}{(z-i)(z+i)^3} \, dz \]  

(27)

where $\gamma$ consists of the real axis and a semicircle at infinity in the upper half plane. The integrand has a triple pole at $z = i$ inside the contour $\gamma$ and so we need to pick up the term proportional to $(x - i)^2$ in the Taylor series expansion of $1/(x + i)^3$ about $z = i$. This is given by

\[ \frac{d^2}{dz^2} \frac{1}{(z+i)^3} \bigg|_{z=i} = \frac{1}{2} \frac{12}{(z+i)^5} \bigg|_{z=i} = -\frac{3i}{16} \]  

(28)

Thus we get

\[ I = 2\pi i \left( -\frac{3i}{16} \right) = \frac{3\pi}{8} \]  

(29)

d) We have

\[ I = \int_{-\infty}^{\infty} \frac{\cos kx \, dx}{(x-a)^2 + b^2} = \text{Re} \oint_{\gamma} \frac{e^{ikz}}{(z-a)^2 + b^2} \, dz \]  

(30)

where $\gamma$ consists of the real axis and a semicircle at infinity in the upper half plane. The integrand has a pole inside the region delimited by $\gamma$ at $z = a + ib$ with residue $\exp[k(ia - b)]/(2ib)$. From the theorem of residues we obtain

\[ \oint_{\gamma} \frac{e^{ikz}}{(z-a)^2 + b^2} \, dz = 2\pi i \frac{e^{k(ia-b)}}{2ib} = \frac{\pi e^{ika} e^{-kb}}{b} \]  

(31)

Taking the real part we get

\[ I = \frac{\pi e^{-kb} \cos ka}{b} \]  

(32)

e)  

\[ I = \int_{0}^{\infty} \frac{x \sin x \, dx}{x^2 + 1} = \frac{1}{2} \int_{-\infty}^{\infty} \frac{x \sin x \, dx}{x^2 + 1} = \frac{1}{2} \text{Im} \int_{-\infty}^{\infty} \frac{xe^{ix} \, dx}{x^2 + 1} = \frac{1}{2} \text{Im} \int_{-\infty}^{\infty} \frac{xe^{ix} \, dx}{x^2 + 1} = 0 \]  

(33)
where $\gamma$ consists of the real axis and a semicircle at infinity in the upper half plane. The integrand has a pole inside the region delimited by $\gamma$ at $z = i$ with residue $a/(2e)$. From the theorem of residues we obtain

$$
\oint \frac{ze^{iz}}{z^2 + 1} \, dz = 2\pi i \frac{a}{2e}
$$

Taking $(1/2)$ times the imaginary part we finally get

$$
I = \frac{\pi}{2e}
$$

f) $I = \int_{-\infty}^{\infty} \cos x \, dx = \Re \int_{-\infty}^{\infty} e^{ix} \, dx$

Close with a semicircle at infinity in the upper complex plane getting a contour $\gamma$ and

$$
I = \Re \oint_{\gamma} \frac{e^{ix}}{(z^2 + a^2)(z^2 + b^2)} \, dz
$$

The integral picks up the residues at $z = ia$ and $z = ib$. Thus we get

$$
I = \Re 2\pi i \left( \frac{e^{-a}}{2ia(-a^2 + b^2)} + \frac{e^{-b}}{2ib(a^2 - b^2)} \right) = \frac{\pi}{a^2 - b^2} \left( \frac{e^{-b}}{b} - \frac{e^{-a}}{a} \right)
$$

(Notice that the contour integral is real, as one would expect from the fact that the integral with $\sin x$ replacing $\cos x$ vanishes by symmetry.)

For $a = b$ we could calculate the integral with a double pole in the integrand, or we can take the limit of the result for $b \to a$. Let us follow this second procedure and set

$$
a = c + \delta, \quad b = c - \delta
$$

with the idea that we will take the limit $\delta \to 0$ and then identify back $c$ with $a$. It is convenient to rewrite $I$ as

$$
I = \frac{\pi}{ab(a + b)} \frac{ae^{-b} - be^{-a}}{a - b}
$$

Substituting from Eq. 39 we get

$$
I = \frac{\pi}{2c(c^2 - \delta^2)} \frac{(c + \delta)e^{-c+\delta} - (c - \delta)e^{-c-\delta}}{2\delta}
$$

$$
= \frac{\pi e^{-c}}{2c(c^2 - \delta^2)} \frac{(c + \delta)e^\delta - (c - \delta)e^{-\delta}}{2\delta}
$$
We expand the exponentials to first order in $\delta$ and in the first denominator we neglect $\delta^2$, which contributes to higher order, to obtain

$$I = \frac{\pi e^{-c}}{2c^3} \left( (c + \delta)(1 + \delta) - (c - \delta)(1 - \delta) \right) = \frac{\pi e^{-c}(c + 1)}{2c^3}$$

or, with $c = a$,

$$I = \frac{\pi e^{-a}(a + 1)}{2a^3}$$