PY 501 - Mathematical Physics
Assignment 2 - September 17, 2020.
To be completed by the end of the day on Thursday, September 24, 2020.

Do all four problems, however see the note below about problems 3 and 4. Each correct solution will be given 25 points, with points taken away for errors (or for absence of explanation) according to the severity of the error.

Problems 3 and 4 ask that one calculates some integrals, 8 for problem 3 and 6 for problem 4. Calculating them all is time consuming, so you only have to return the evaluation of 4 integrals of your own choosing for problem 3, and of 3 integrals, again of your own choosing, for problem 4. You may want to calculate also the other integrals or some of them as an exercise, and you will be able to check your work when I post the solutions, but only submit the requested numbers of evaluations for problems 3 and 4.

Problem 1:
The finite sum

\[ f(x) = \sum_{n=0}^{N} e^{inx} \]  

is a geometric progression and can easily be summed

\[ f(x) = \sum_{n=0}^{N} (e^{ix})^n = \frac{1 - e^{(N+1)x}}{1 - e^{ix}} \]  

Use this result and Euler’s formula to prove the identity

\[ f(x) = \sum_{n=0}^{N} \cos nx = \frac{\sin \frac{1}{2}(N + 1)x}{\sin \frac{1}{2}x} \cos \frac{Nx}{2} \quad (x = \text{real}) \]

What is the corresponding sum involving \( \sin nx \)?
Problem 2:
Consider the two functions
\[ u(x, y) = \frac{x(1 + x) + y^2}{(1 + x)^2 + y^2} \]  \hspace{1cm} (4)
\[ v(x, y) = \frac{y}{(1 + x)^2 + y^2} \]  \hspace{1cm} (5)

Can \( u(x, y) \) and \( v(x, y) \) be the real and imaginary part of an analytic function \( f(z) \)?

Answer either by checking whether \( u \) and \( v \) satisfy the Cauchy-Riemann conditions or by any other method of your choice.

If the answer is yes, what is \( f(z) \)?

Problem 3:
Using the Cauchy theorem, Cauchy Integral formula, or their consequences, evaluate the following integrals, all taken around the circle \( |z| = 2 \):

a) \( \oint \frac{\cos z}{z} \, dz \)  

b) \( \oint \frac{\sin z}{z} \, dz \)  

c) \( \oint \frac{e^z}{z-1} \, dz \)  

d) \( \oint \frac{2z^2 + 3z - 1}{z^2 + 1} \, dz \)  

e) \( \oint \frac{2z}{z^2 - 9} \, dz \)  

f) \( \oint \frac{\sin z}{z^2} \, dz \)  

g) \( \oint \frac{e^z}{(z-1)^7} \, dz \)  

h) \( \oint \frac{\sin z}{z^2} \, dz \)

Problem 4:
Establish the following results:

a) \( \int_{-\infty}^{\infty} \frac{dx}{x^2 + a^2} = \frac{\pi}{a\sqrt{2}} \)  
b) \( \int_{0}^{\infty} \frac{x^2}{x^4 + 1} \, dx = \frac{\pi}{6} \)

c) \( \int_{-\infty}^{\infty} \frac{dx}{(x^2 + 1)^2} = \frac{3\pi}{8} \)  
d) \( \int_{-\infty}^{\infty} \frac{\cos kx \, dx}{(x-a)^2 + b^2} = \frac{\pi}{b} \) \( e^{-kb} \cos ka \) \( (k > 0, \ b > 0) \)

e) \( \int_{0}^{\infty} \frac{\sin x \, dx}{x^2 + 1} = \frac{\pi}{2e} \)  
f) \( \int_{-\infty}^{\infty} \frac{\cos x \, dx}{(x^2 + a^2)(x^2 + b^2)} = \frac{\pi}{a^2 - b^2} \left( \frac{e^{-b}}{b} - \frac{e^{-a}}{a} \right) \) \( (a \neq b) \)

What is the result of f) if \( a = b \)?