Midterm exam:

Do all three problems. After a base of 10 points to make the maximum equal to 100, each correct solution will be given 30 points, with points taken away for errors (or for absence of explanation) according to the severity of the error.

Problem 1:
Prove the identity
\[
\int_{-\infty}^{\infty} \frac{e^{ax}}{1 + e^x} \, dx = \frac{\pi}{\sin \pi a} \quad (0 < |a| < 1) \tag{1}
\]
by expressing the integral in terms of a contour integral over the rectangle with sides along the lines \(y = 0, y = 2\pi, x = -R, \) and \(x = R,\) sending then \(R \to \infty.\)

Solution
This was problem 2 in assignment 3. Please see the posted solutions to assignment 3.

Problem 2:
The equation
\[
w = \frac{1}{z} \tag{2}
\]
defines a conformal mapping from the complex \(z\)-plane to the complex \(w\)-plane in any finite region in the \(z\)-plane which excludes the origin.

A) Prove that straight lines in the \(z\)-plane which do not pass through the origin are mapped into circles in the \(w\)-plane which pass through the origin.

B) The real axis in the \(z\)-plane is obviously mapped into the real axis in the \(w\)-plane. Consider the line defined by
\[
z = 2 + te^{i\phi} \quad (3)
\]
with $-\infty < t < \infty$ and fixed $\phi \neq 0$, which forms an angle $\phi$ with the real axis, and show that it is mapped into a circle which intersects the positive real axis in the $w$-plane at an angle $\phi$.

Solution

A) Let us express $z$ and $w$ in terms of their real and imaginary parts as

$$z = x + iy$$
$$w = u + iv$$

From Eq. 2 we derive the equation

$$z = x + iy = \frac{1}{w} \frac{w^*}{ww^*} = \frac{u - iv}{u^2 + v^2}$$

from which, equating the real and imaginary parts, we obtain

$$x = \frac{u}{u^2 + v^2}$$
$$y = -\frac{v}{u^2 + v^2}$$

The equation of a straight line in the $z$-plane is

$$ax + by + c = 0$$

where $c \neq 0$, because the line does not pass through the origin, and at least one of $a$ and $b$ not equal to zero. Substituting from Eqs. 7, 8, we find the equation of the image in the $w$-plane:

$$\frac{au}{u^2 + v^2} - \frac{bv}{u^2 + v^2} + c = 0$$

or

$$u^2 + v^2 + \frac{a}{c} u - \frac{b}{c} v = 0$$

which is the equation of a circle. Since the equation is satisfied with $u = v = 0$, the circle passes through the origin.

B) Equation 3

$$z = x + iy = 2 + t \cos \phi + it \sin \phi$$
implies the equations

\begin{align*}
x &= 2 + t \cos \phi \\
y &= t \sin \phi
\end{align*}  \quad (13)

which give a parametric description of a straight line going through the point \((1, 0)\) at an angle \(\phi\). By eliminating the parameter \(t\) we find

\[x \sin \phi - y \cos \phi - 2 \sin \phi = 0 \quad (15)\]

for the equation of the line. From the result obtained in part A, with \(a = \sin \phi, \ b = -\cos \phi, \ c = -2 \sin \phi\), we see that the equation of the corresponding circle in the \(w\)-plane is

\[u^2 + v^2 - \frac{1}{2} u - \frac{\cos \phi}{2 \sin \phi} v = 0 \quad (16)\]

By setting \(v = 0\) we see that the circle crosses the real axis at \(u = 0\) and \(u = 1/2\). By differentiating Eq. 16 we get

\[2u \, du + 2v \, dv - \frac{1}{2} \, du - \frac{\cos \phi}{2 \sin \phi} \, dv = 0 \quad (17)\]

which, with \(u = 1/2, v = 0\) gives

\[\frac{dv}{du} = \frac{\sin \phi}{\cos \phi} \quad (18)\]

which shows that the tangent to the circle forms an angle \(\phi\) with the real axis.

Equivalently, we could have taken the differential of Eq. 2

\[dw = -\frac{dz}{z^2} \quad (19)\]

which for \(z = 2\) gives

\[dw = -\frac{dz}{4} \quad (20)\]

or

\[du + udv = -\frac{dx + udy}{4} \quad (21)\]
\[ du = \frac{\omega_0^2}{4} dx \]  
(22)

\[ dy = \frac{1}{4} dy \]  
(23)

which again shows

\[ \frac{dv}{du} = \frac{dy}{dx} \]  
(24)

**Problem 3:**

Consider the forced harmonic oscillator equation

\[ \frac{d^2 y(t)}{dt^2} + \omega_0^2 y(t) = F(t) \]  
(25)

and assume that the system is at rest with \( y(t) = 0, \frac{dy(t)}{dt} = 0 \) and \( F(t) = 0 \) for \(-\infty < t \leq 0\).

Use the method of variation of the constants to find the solution of the equation with the following driving force:

\[ F(t) = \sin \omega t \quad \text{for } t \geq 0 \]  
(26)

In particular, show that if \( \omega = \omega_0 \) the amplitude of the oscillation grows without bound.

**Important note:** you must find the explicit form of the solution; you may not leave it in an implicit integral representation.

**Hint:** The solution of this problem is, in principle, straightforward, but the algebra of the trigonometric functions can become quite involved. I found it easier to solve the problem with a driving force \( \exp(i\omega t) \), taking at the end the imaginary part of the solution. Also I found convenient to use \( \exp(i\omega_0 t) \) and \( \exp(-i\omega_0 t) \) as the two solutions of the homogeneous equation.

I recommend that you do the algebra carefully and without skipping lines. Whenever I tried to skip a line I ended up making a mistake.

(Recap from the lecture notes: Given two independent solutions \( y_1(x), y_2(x) \) of linear, homogeneous ODE \( y'' + P(x)y' + Q(x)y = 0 \), a solution \( y(x) \) of the non-homogeneous equation \( y'' + P(x)y' + Q(x)y = R(x) \) can be found by the method of variation of constants, which gives \( y(x) \) in the form

\[ y(x) = c_1(x)y_1(x) + c_2(x)y_2(x) \]  
(27)
with

\[
\begin{align*}
c_1(x) &= -\int_{x_1}^{x} \frac{R(x')}{W(x')} \, dx' \\
c_2(x) &= \int_{x_2}^{x} \frac{R(x')}{W(x')} \, dx'
\end{align*}
\]  

where

\[
W = y_1(x)y_2'(x) - y_2(x)y_1'(x)
\]

is the Wronskian of the two solutions.

**Solution**

We will consider \( y(t) \) as the imaginary part of the solution obtained with a driving force

\[
F(t) = e^{i\omega t}
\]

We take

\[
\begin{align*}
y_1(t) &= e^{i\omega_0 t} \\
y_2(t) &= e^{-i\omega_0 t}
\end{align*}
\]

as the two solutions of the homogeneous equation. The Wronskian of the two solutions is

\[
W(t) = -2i\omega_0
\]

Thus, from the equations given above, we find the following:

\[
y(t) = -e^{i\omega_0 t} \int_0^t \frac{e^{i\omega_0 t'} e^{-i\omega_0 t'}}{-2i\omega_0} \, dt' + e^{-i\omega_0 t} \int_0^t \frac{e^{i\omega_0 t'} e^{i\omega_0 t'}}{-2i\omega_0} \, dt' =
\]

\[
-\frac{e^{i\omega_0 t} e^{i(\omega_0 - \omega_0)t} - 1}{-2i\omega_0(\omega - \omega_0)} + \frac{e^{-i\omega_0 t} e^{i\omega_0 t} - 1}{-2i\omega_0(\omega + \omega_0)} =
\]

\[
\frac{e^{i\omega_0 t}}{2\omega_0} \left( -\frac{1}{\omega - \omega_0} + \frac{1}{\omega + \omega_0} \right) + \frac{e^{-i\omega_0 t}}{2\omega_0} \left( -\frac{1}{\omega - \omega_0} + \frac{1}{\omega + \omega_0} \right) =
\]

\[
\frac{e^{i\omega_0 t}}{\omega_0^2 - \omega^2} + \frac{e^{-i\omega_0 t}}{2\omega_0(\omega - \omega_0)} - \frac{e^{-i\omega_0 t}}{2\omega_0(\omega + \omega_0)}
\]

Check:
The last two terms in Eq. 35 are solutions of the homogeneous equation, so we only need to check that the first term

$$\tilde{y} = \frac{e^{i\omega t}}{\omega_0^2 - \omega^2}$$

(36)

satisfies the inhomogeneous equation. We have

$$\tilde{y}'' = \frac{-\omega^2 e^{i\omega t}}{\omega_0^2 - \omega^2}$$

(37)

and thus

$$\tilde{y}'' + \omega_0^2 \tilde{y} = \frac{-\omega^2 e^{i\omega t}}{\omega_0^2 - \omega^2} + \frac{\omega^2 e^{i\omega t}}{\omega_0^2 - \omega^2} = e^{i\omega t}$$

(38)

Moreover the boundary conditions $y(0) = y'(0) = 0$ are also satisfied, as one can easily check.

By taking the imaginary part of $y(t)$ we find the solution to our problem:

$$y(t) = \frac{\sin \omega t}{\omega_0^2 - \omega^2} + \frac{\sin \omega_0 t}{2\omega_0(\omega - \omega_0)} + \frac{\sin \omega_0 t}{2\omega_0(\omega + \omega_0)} = \frac{\omega_0 \sin \omega t - \omega \sin \omega_0 t}{\omega_0(\omega_0^2 - \omega^2)}$$

(39)

Finally, with $\omega = \omega_0$ Eq. 35 reduces to

$$y(t) = -e^{i\omega_0 t} \int_0^t \frac{1}{-2i\omega_0} dt' + e^{-i\omega_0 t} \int_0^t e^{2i\omega_0 t'} \frac{1}{-2i\omega_0} dt' =$$

$$\frac{te^{i\omega_0 t} + e^{i\omega_0 t} - e^{-i\omega_0 t}}{2\omega_0^2} = \frac{t(-i \cos \omega_0 t + \sin \omega_0 t)}{2\omega_0} + \frac{\sin \omega_0 t}{2\omega_0^2}$$

(40)

and it is clear that both the real and imaginary part of $y(t)$ grow without bound. Taking the imaginary part we get

$$y(t) = -\frac{t \cos \omega_0 t}{2\omega_0} + \frac{\sin \omega_0 t}{2\omega_0}$$

(41)

a result which one could have also obtained by taking the limit of Eq. 39 for $\omega \to \omega_0$.

The behavior of the solution for three values of the ratio $\omega/\omega_0$ is illustrated in the next page.
Figure 1: Behavior of the solution for $\frac{\omega}{\omega_0} = 0.5$ (blue dashed line), $\frac{\omega}{\omega_0} = 0.9$ (green dashed line), $\frac{\omega}{\omega_0} = 0.99$, practically indistinguishable from $\omega = \omega_0$, red solid line.