1 Problem 1

A
From the problem, we can know that the surface $\Sigma$ is:

$$x^2 + y^2 + z^2 = R^2 \quad (1)$$

Because of the symmetry, the surface integral of the total sphere is twice of the surface integral of the top half sphere ($z > 0$):

$$I = 2I' = 2\int_{\text{top}} \vec{v}(x,y,z) \cdot \hat{n} d\sigma \quad (2)$$

$$= 2\int_{\text{top}} (x^3, y^3, z^3) \frac{1}{R}(x, y, z) d\sigma \quad (3)$$

$$= 2\int_{\text{top}} (x^4 + y^4 + z^4)/R \cdot R^2 \sin \phi d\phi d\theta \quad (4)$$

$$= 2R^5 \int_0^{2\pi} \int_0^{\frac{\pi}{2}} (\sin^4 \phi \cos^4 \theta + \sin^4 \phi \sin^4 \theta + \cos^4 \phi) \sin \phi d\phi d\theta \quad (5)$$

$$= 2R^5 \left[ \int_0^{2\pi} \int_0^{\frac{\pi}{2}} \sin^5 \phi \cos^4 \theta d\phi d\theta + \int_0^{2\pi} \int_0^{\frac{\pi}{2}} \sin^5 \phi \sin^4 \theta d\phi d\theta + \int_0^{2\pi} \int_0^{\frac{\pi}{2}} \sin \phi \cos^4 \theta d\phi d\theta \right] \quad (6)$$

$$= 12\pi R^5 \frac{2\pi}{5} \frac{2\pi}{5} \frac{2\pi}{5} \quad (7)$$

$$= \frac{12\pi R^5}{5} \quad (8)$$

B
Using Gauss' theorem,

$$I = \int_{\Sigma} \vec{v}(x,y,z) \cdot \hat{n} d\sigma \quad (9)$$

$$= \int_V \nabla \cdot \vec{v} dV \quad (10)$$

$$= 3 \int_V (x^2 + y^2 + z^2) dV \quad (11)$$

$$= 3 \int_0^{2\pi} \int_0^{\pi} \pi d\theta \int_0^R r^2 r^2 dr \quad (12)$$

$$= 3 \cdot 2\pi \cdot 2 \cdot \frac{R^5}{5} \quad (13)$$

$$= \frac{12\pi R^5}{5} \quad (14)$$

C
Using the formula for a surface integral,

\[ I = \int_{\text{top}} \omega(x,y,z) \cdot \mathbf{n} \, d\sigma + \int_{\text{bottom}} \omega(x,y,z) \cdot \mathbf{n} \, d\sigma \]  

(15)

\[ = \int_{\text{top}} (y^2 z^2, x^2 z^2, x^2 y^2) \frac{1}{R} \, d\sigma + \int_{\text{bottom}} \omega(x,y,z) \cdot \mathbf{n} \, d\sigma \]  

(16)

\[ = \int_{\text{top}} (xy^2 z^2, x^2 y z^2, x^2 y^2 z) / R \cdot R^2 \sin \phi \, d\phi d\theta + \int_{\text{bottom}} \omega(x,y,z) \cdot \mathbf{n} \, d\sigma \]  

(17)

\[ = R^6 \int_0^{2\pi} \int_0^{\pi/2} (\sin^4 \phi \cos^2 \phi \sin^2 \theta \cos \theta + \sin^4 \phi \cos^2 \phi \sin \theta \cos^2 \phi \cos \theta \cos^2 \theta) \, d\phi d\theta \]  

(18)

\[ + \int_{\text{bottom}} \omega(x,y,z) \cdot \mathbf{n} \, d\sigma \]  

(19)

\[ = R^6 (0 + 0 + \frac{\pi}{24}) - R^6 \frac{\pi}{24} \]  

(20)

\[ = 0 \]  

(21)

Using Gauss' theorem,

\[ I = \oint_{\Sigma} \omega(x,y,z) \cdot \mathbf{n} \, d\sigma \]  

(22)

\[ = \int_{V} \nabla \cdot \omega \, dV \]  

(23)

\[ = 3 \int_{V} 0 \, dV \]  

(24)

\[ = 0 \]  

(25)

2 Problem 2

The integral is dependent of the path. We could assume:

\[ z = x + yi \]  

(26)

\[ f(z) = \frac{1}{1 - z^2} \]  

(27)

Then, we could indicate two paths in the z-plan. Obviously, the function \( f(z) \) has two poles \( z = \pm 1 \).
• If the two paths $C_1$ and $C'_1$ close the pole $z = 1$, which is shown in Figure 1, the difference could be calculated:

$$\Delta I_1 = \int_{-i,C_1}^{2+i} \frac{dz}{1-z^2} - \int_{-i,C'_1}^{2+i} \frac{dz}{1-z^2}$$

$$= \int_{-i,C_1}^{2+i} \frac{dz}{1-z^2} + \int_{-i,C'_1}^{-i} \frac{dz}{1-z^2}$$

$$\quad = \oint_{\Gamma_1} \frac{dz}{1-z^2}$$

$$= 2\pi i \text{Res}(f(z = 1)) = 2\pi i \lim_{z \to 1} \frac{1}{z-1} \frac{1}{1-z^2}$$

$$= -\pi i$$
• If the two paths $C_2$ and $C'_2$ close the pole $z = -1$, which is shown in Figure 1, the difference could be calculated:

$$
\Delta I_2 = \int_{-i,C_2}^{2+i} \frac{dz}{1-z^2} - i \int_{-i,C'_2}^{2+i} \frac{dz}{1-z^2} 
$$

(33)

$$
= \int_{-i,C_2}^{2+i} \frac{dz}{1-z^2} + \int_{i,C_2}^{-i} \frac{dz}{1-z^2} 
$$

(34)

$$
= \oint_{\Gamma_2} \frac{dz}{1-z^2} 
$$

(35)

$$
= 2 \pi i \text{Res}(f(z=-1)) = 2 \pi i \lim_{z \to -1} (z+1) \frac{1}{1-z^2} 
$$

(36)

$$
= \pi i 
$$

(37)

• If the two paths $C_3$ and $C'_3$ close the two poles, which is shown in Figure 1, the difference could be calculated:

$$
\Delta I_3 = \int_{-i,C_3}^{2+i} \frac{dz}{1-z^2} - i \int_{-i,C'_3}^{2+i} \frac{dz}{1-z^2} 
$$

(38)

$$
= \int_{-i,C_3}^{2+i} \frac{dz}{1-z^2} + \int_{i,C_3}^{-i} \frac{dz}{1-z^2} 
$$

(39)

$$
= \oint_{\Gamma_3} \frac{dz}{1-z^2} 
$$

(40)

$$
= 2 \pi i [\text{Res}(f(z = 1)) + \text{Res}(f(z = -1))] = 2 \pi i (-\frac{1}{2} + \frac{1}{2}) 
$$

(41)

$$
= 0 
$$

(42)

• If the two paths $C_4$ and $C'_4$ do not close the pole, which is shown in Figure 2, via the Cauchy theorem, the difference could be calculated:

$$
\Delta I_4 = \int_{-i,C_4}^{2+i} \frac{dz}{1-z^2} - i \int_{-i,C'_4}^{2+i} \frac{dz}{1-z^2} 
$$

(43)

$$
= \int_{-i,C_4}^{2+i} \frac{dz}{1-z^2} + \int_{i,C'_4}^{-i} \frac{dz}{1-z^2} 
$$

(44)

$$
= \oint_{\Gamma_4} \frac{dz}{1-z^2} 
$$

(45)

$$
= 0 
$$

(46)
3 Problem

We could rewrite the equation as:

\[ y'' - \frac{y}{x^4} = 0 \]  \hspace{1cm} (47)

\[ Q(x) = \frac{1}{x^4}, \text{ which is not analytical at the original, so that the equation has no solutions of the Frobenius type.} \]

In order to solve the equation, we could set:

\[ z = \frac{1}{x} \]  \hspace{1cm} (48)

Therefore,

\[ \frac{d}{dx} = \frac{dz}{dx} \frac{d}{dz} \]  \hspace{1cm} (49)

\[ \frac{dz}{dx} = -\frac{1}{x^2} \]  \hspace{1cm} (50)

Then, We could rewrite the equation as:

\[ x^4(-\frac{1}{x^2} \frac{d}{dz})(-\frac{1}{x^2} \frac{d}{dz}y) - y = 0 \]  \hspace{1cm} (51)

\[ x^2 \frac{d}{dz} \frac{1}{x^2} \frac{d}{dz}y - y = 0 \]  \hspace{1cm} (52)

\[ \frac{1}{x^2} \frac{d}{dz}x^2 \frac{d}{dz}y - y = 0 \]  \hspace{1cm} (53)

\[ \frac{d}{dz}z^2 \frac{d}{dz}y - z^2y = 0 \]  \hspace{1cm} (54)

Based on the solution given by the problem, we could define \( y(z) = \frac{w(z)}{z} \), so that:

\[ \frac{d}{dz}z^2 \left( \frac{w'}{z} - \frac{w}{z^2} \right) - zw = 0 \]  \hspace{1cm} (55)

Simplify it, we could get:

\[ w'' - w = 0 \]  \hspace{1cm} (56)

Obviously, solutions for equation 31 is:

\[ w(z) = Acosh(z) + Bsinh(z) \]  \hspace{1cm} (57)

So that,

\[ y(z) = \frac{w(z)}{z} = \frac{A}{z} \cosh(z) + \frac{B}{z} \sinh(z) \]  \hspace{1cm} (58)

\[ y(x) = Axcosh(1/x) + Bxsinh(1/x) \]  \hspace{1cm} (59)
4 Problem 4

A
Using the equation (68) in the note "Partial derivative differential equations I", we could know that:

\[
\frac{dx}{v(x)} = dt \tag{60}
\]

\[
\frac{dx}{c x^2 + l^2} = dt \tag{61}
\]

\[
x^2 + 2l^2 \frac{dx}{x^2 + l^2} = cdt \tag{62}
\]

\[
\frac{l^2}{x^2 + l^2} dx = cdt \tag{63}
\]

\[
\int 1dx + l \int \frac{1}{1 + (\frac{x}{l})^2} d\frac{x}{l} = \int cdt \tag{64}
\]

\[
x + l \arctan(\frac{x}{l}) = ct \tag{65}
\]

If we choose the negative sign, the characteristics will be:

\[
x + l \arctan(\frac{x}{l}) = -ct \tag{66}
\]

B
When \( t \to \infty, x >> l \). Therefore, \( \arctan(\frac{x}{l}) \to \frac{\pi}{2} \), and the equation (40) could be written as:

\[
ct = x + \frac{\pi}{2} l \tag{67}
\]

\[
x = ct - \frac{\pi}{2} l \tag{68}
\]

So that, it would lag \( \frac{\pi}{2} l \).

C
For the trajectories of the perturbation:

\[
t_1 = x + \arctan(x) \tag{69}
\]

For the ray of light:

\[
t_2 = x \tag{70}
\]

The space-time diagram is drawn in the Figure 2.
Figure 2: Space-time diagram