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1. PROBLEM 1

1.1. Problem 1A

Given

\[- \frac{d^2 f(x)}{dx^2} = b(x) \]  \hspace{1cm} (1)

We can integrate both sides

\[- \int_{-L/2}^{L/2} \frac{d^2 f(x)}{dx^2} dx = f'(L/2) - f'(L/2) = 0 \]  \hspace{1cm} (2)

due to periodic boundary conditions, which implies

\[\int_{-L/2}^{L/2} b(x) \, dx = 0 \]  \hspace{1cm} (3)

Thus there only exists a solution \( f \) for functions \( b \) with average value \( \langle b \rangle = 0 \).

1.2. Problem 1B

Given

\[- \frac{d^2 \tilde{f}(x)}{dx^2} = b(x) + c \]  \hspace{1cm} (4)

we look for a Green’s function that satisfies

\[- \int_{-L/2}^{L/2} \frac{d^2 G(x, y)}{dx^2} b(y) \, dy = b(x) + c \]  \hspace{1cm} (5)

Try a Green’s function satisfying

\[- \frac{d^2 G(x, y)}{dx^2} = \delta(x - y) + c_2 \]  \hspace{1cm} (6)

In order for \( G(x, y) \) to be periodic, it must satisfy
\[- \int_{-L/2}^{L/2} \frac{d^2 G(x,y)}{dx^2} \, dx = \frac{dG}{dx}(L/2,y) - \frac{dG}{dx}(-L/2,y) = 0 \quad (7)\]

so from this we can deduce that the constant \( c_2 \) is equal to \(- \frac{1}{L}\).

Now we look for the general solution to (6), which may be different for \( x < y \) and \( x > y \).

\[ G(x,y) = \begin{cases} A_1 + B_1 x + \frac{1}{2L} x^2 & x < y \\ A_2 + B_2 x + \frac{1}{2L} x^2 & x > y \end{cases} \quad (8) \]

Now we use the periodic boundary conditions of \( G(x,y) \) to narrow down the space of solutions. First, we require that the two pieces of \( G \) agree at the point \( y \) so that \( G \) is continuous:

\[ A_1 + B_1 y + \frac{1}{2L} y^2 = A_2 + B_2 y + \frac{1}{2L} y^2 \quad (9)\]

So that

\[ A_2 - A_1 = (B_1 - B_2)y \quad (10)\]

Now use that fact that \( G'(L/2,y) = G'(-L/2,y) \)

\[ B_1 + \frac{1}{L}(-L/2) = B_2 + \frac{1}{L}(L/2) \quad (11)\]

which gives us that \( B_1 - B_2 = 1 \) and \( A_2 - A_1 = y \).

Finally, assert that \( G(-L/2,y) = G(L/2,y) \), so

\[ A_1 + B_1(-L/2) + \frac{1}{2L} \left( \frac{L^2}{4} \right) = A_1 + y + (B_1 - 1)(L/2) + \frac{1}{2L} \left( \frac{L^2}{4} \right) \quad (12)\]

and thus \( B_1 = \frac{1}{2} - \frac{y}{L} \). Altogether, we get

\[ G(x,y) = \begin{cases} A_1 + \left( \frac{1}{2} - \frac{y}{L} \right)x + \frac{1}{2L} x^2 & x < y \\ A_1 + y - \left( \frac{1}{2} + \frac{y}{L} \right)x + \frac{1}{2L} x^2 & x > y \end{cases} \quad (13)\]
The symbol $A_1$ here is an arbitrary function of $y$, which we choose so that $G(x,y)$ is symmetric:

$$G(x,y) = \begin{cases} 
A_0 + \frac{y^2}{2L} - \frac{y}{2} + (\frac{1}{2} - \frac{y}{L})x + \frac{1}{2L}x^2 & x < y \\
A_0 + \frac{y^2}{2L} + \frac{y}{2} - (\frac{1}{2} + \frac{y}{L})x + \frac{1}{2L}x^2 & x > y 
\end{cases}$$

(14)

There are a couple things to notice here. First $\frac{dG(x,y)}{dx}$ is discontinuous at $x = y$, as we expect since there is a delta function in $G''$. Second, the function $G(x,y)$ is only defined up to a constant, since we have no more conditions to determine the value of $A_0$.

1.3. Problem 1C

We can find the constant $c$ using equations (5) and (6)

$$-\frac{d^2\tilde{f}(x)}{dx^2} = \int_{-L/2}^{L/2} \delta(x - y) b(y) dy - \frac{1}{L} \int_{-L/2}^{L/2} b(y) dy$$

(15)

$$= b(x) - \langle b \rangle.$$  \hspace{2cm} (16)

Thus the constant $c$ is equal to $-\langle b \rangle$. 

2. PROBLEM 2

We look at the saw-tooth function. To simplify the algebra given by choosing $a = T = 1$.

2.1. Problem 2A

We do the Fourier series representation of this function

$$x(t) = A_0 + \sum_{n=1}^{\infty} \left[ A_n \cos(2n\pi t) + B_n(2n\pi t) \right]$$

(17)

Due to the periodic nature of our function, we look between the domain $[0,1]$.

We start off the simplest case $A_0$ which has to be analyzed

$$A = \int_0^1 t \, dt$$

$$= \frac{1}{2}$$

(18) (19)

We look at the other terms of

$$A_n = 2 \int_0^1 t \cos(2n\pi t) \, dt$$

(20)

We have to do an integration by parts to find

$$= 2 \left[ \frac{t \sin(2n\pi t)}{2n\pi} \right]_0^1 - \int_0^1 \frac{\sin(2n\pi t)}{2n\pi} \, dt$$

$$= 2 \frac{\cos(2n\pi t)}{4n^2\pi^2} \bigg|_0^1 = 0$$

(21) (22)

We find that the cosine series terms goes to zero. Now we look at the sine series terms

$$B_n = 2 \int_0^1 t \sin(2n\pi t) \, dt$$

(23)

We get the integration by parts as before to find

$$B_n = \frac{-1}{n\pi}$$

(24)

Our final expression becomes

$$x(t) = \frac{1}{2} - \sum_{j=1}^{\infty} \frac{\sin(2j\pi t)}{j\pi}$$

(25)

The normalized saw-tooth function can also be represented in the following compact notation using the floor function

$$x(t) = t - \lfloor t \rfloor$$

(26)
2.2. Part 2B

We plot our results for different values of $n$.

\[ \sum_{j=1}^{n} \sin(j \pi t) \]

FIG. 1: We plot the cumulative sum for our series upto the $n = 4$ term.

FIG. 2: We plot the cumulative sum for our series upto the $n = 50$ term. Here we clearly see the convergence towards the saw-tooth behaviour.
3. PROBLEM 3

3.1. Problem 3A

We do the Fourier transform of the following function given by

\[ f(x) = e^{-\lambda x^2} \cos(\beta x) \]  
\[ = \frac{1}{2} \left[ e^{-\lambda x^2} (e^{i\beta x} + e^{-i\beta x}) \right] \]  

The Fourier transform is given by

\[ F(x) = \frac{1}{2\sqrt{2\pi}} \int_{-\infty}^{\infty} \left[ e^{-\lambda x^2} (e^{i\beta x} + e^{-i\beta x}) \right] e^{ikx} dx \]  

This Fourier transform is a Gaussian integral which we can solve using this integral

\[ \int_{-\infty}^{\infty} \exp(-ax^2 + bx) dx = \sqrt{\frac{\pi}{a}} e^{\frac{b^2}{4a}} \]  

We find our final integral to be

\[ = \frac{1}{2\sqrt{2\lambda}} \left( \exp \left[ -\frac{\beta^2 + 2k\beta + k^2}{4\lambda} \right] + \exp \left[ -\frac{\beta^2 - 2k\beta + k^2}{4\lambda} \right] \right) \]
3.2. Problem 3B

We compute the Fourier transform of the Fourier integral of the following function

\[ f(x) = \frac{\cos(\beta x)}{x^4 + a^4} = \frac{e^{i\beta x} + e^{-i\beta x}}{2(x^4 + a^4)} \quad (32) \]

Here we have written the cosine terms using complex exponentials.

The Fourier transform is given by

\[ F(k) = \frac{1}{2\sqrt{2\pi}} \int_{-\infty}^{\infty} \frac{e^{i\beta x} + e^{-i\beta x}}{2(x^4 + a^4)} \exp(ikx) \, dx \quad (34) \]

We can do this as a contour integral.

\[ F(k) = \frac{1}{2\sqrt{2\pi}} \oint \frac{\exp(ix(k + \beta)) + \exp(ix(k - \beta))}{(z - z_1)(z - z_2)(z - z_3)(z - z_4)} \quad (35) \]

Our choice of the contour is a semi-circle in the upper half of the complex plane. Our contour has 4 simple poles.

\[ z_1 = ae^{i\pi/4} = a \frac{1 + i}{\sqrt{2}} \quad (36) \]
\[ z_2 = ae^{i3\pi/4} = a \frac{-1 + i}{\sqrt{2}} \quad (37) \]
\[ z_3 = ae^{i5\pi/4} = a \frac{-1 - i}{\sqrt{2}} \quad (38) \]
\[ z_4 = ae^{i7\pi/4} = a \frac{1 - i}{\sqrt{2}} \quad (39) \]

For our choice of the contour only the first two poles contribute to the integral. The contribution along the semi-circular arc goes to zero by Jordan’s lemma. By the residue theorem we find

\[ I = 2\pi i \sum_{z_1, z_2} \text{Residues} \]
\[ = \frac{\pi}{a^3\sqrt{2}} e^{i7\pi/4} \left( e^{k+\beta} + e^{k-\beta} \right) \left( e^{a3\pi/4} + e^{a5\pi/4} \right) \quad (41) \]
3.3. Problem 3C

We analyze the fourier transform of the following piecewise function

\[
f(x) = \begin{cases} 
\cos(k_0 x) & |x| < N\pi/k_0 \\
0 & |x| > N\pi/k_0 
\end{cases}
\]

where \( N \) is any arbitrary integer.

The fourier transforms of \( f(x) \) is found to be

\[
F(k) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} f(x) e^{ikx} \, dx 
\]

\[(43)\]

\[
= \frac{1}{\sqrt{2\pi}} \int_{-N\pi/k_0}^{N\pi/k_0} \cos(k_0 x) e^{ikx} \, dx
\]

\[(44)\]

\[
= \frac{1}{2\sqrt{2\pi}} \int_{-N\pi/k_0}^{N\pi/k_0} e^{i(k_0+k)x} + e^{i(k-k_0)x} \, dx
\]

\[(45)\]

\[
= \frac{1}{2i\sqrt{2\pi}(k^2 - k_0^2)} \left[ (k - k_0)e^{ix(k+k_0)} + (k + k_0)e^{ix(k-k_0)} \right]_{-N\pi/k_0}^{N\pi/k_0}
\]

\[(46)\]

\[
= \frac{1}{2i\sqrt{2\pi}(k^2 - k_0^2)} \left[ (k - k_0) \left( e^{i(k_0+k)N\pi/k_0} - e^{-i(k_0+k)N\pi/k_0} \right) + (k + k_0) \left( e^{i(k-k_0)N\pi/k_0} - e^{-i(k-k_0)N\pi/k_0} \right) \right]
\]

\[(47)\]

\[
= \frac{(k - k_0) \sin((k_0 + k)N\pi/k_0) + (k + k_0) \sin((k - k_0)N\pi/k_0)}{\sqrt{2\pi(k^2 - k_0^2)}}
\]

\[(48)\]

Here, we briefly summarize the above steps. We first expressed \( \cos(k_0 x) \) as complex exponentials. After solving the integral and plugging in the boundary conditions, we group terms and we are able to write our final answer in terms of \( \sin \).
4. **PROBLEM 4**

For times $t > 0$ the equation for the current $i(t)$ is

$$
(L + L') i'(t) + (R + R') i(t) = \epsilon_0
$$

(49)

Taking the Laplace transform of this equation we get

$$
(L + L') (s I(s) - i(0)) + (R + R') I(s) = \frac{\epsilon_0}{s}
$$

(50)

where $i(0)$ is the steady state current before the switch is closed.

For $t < 0$

$$
L i'(t) + R i(t) = \epsilon_0
$$

(51)

In the steady state we can say that $i'(t) = 0$ and so

$$
i(0) = \frac{\epsilon}{R}
$$

(52)

and so we can solve for $I(s)$

$$
I(s) = \frac{\epsilon_0 (1 + \frac{(L+L')}{R} s)}{(L + L') s^2 + (R + R') s}
$$

(53)

All that remains is to perform the inverse Laplace transform to find $i(t)$. This can be done using the Mellin inversion integral

$$
i(t) = \frac{1}{2\pi i} \int_{\gamma-i\infty}^{\gamma+i\infty} e^{tz} I(z) \, dz
$$

(54)

$$
= \frac{1}{2\pi i} \int_{\gamma-i\infty}^{\gamma+i\infty} e^{tz} \frac{\epsilon_0 (1 + \frac{(L+L')}{R} s)}{(L + L') s^2 + (R + R') s} \, dz
$$

(55)

$$
= \frac{\epsilon_0}{2\pi i} \frac{1}{(L + L')} \int_{\gamma-i\infty}^{\gamma+i\infty} e^{tz} \frac{(1 + A z)}{z(z + B)} \, dz
$$

(56)

where $A = \frac{(L+L')}{R}$, $B = \frac{(R+R')}{(L+L')}$, and $\gamma$ in this case may be any positive real number. These integrals can be evaluated by performing the contour integral around the rectangular path $c$, which contains side I going from $\gamma - iH$ to $\gamma + iH$, side II from $\gamma + iH$ to $-W + iH$, side III from $-W + iH$ to $-W - iH$, and side IV from $-W - iH$ back to $\gamma - iH$. When we take $H, W \to \infty$ the contribution from sides II, III, and IV vanishes and we are left with
\[ i(t) = \sum_{res} e^{tz} I(z) \]  \hspace{1cm} (57)

\[ = \frac{\epsilon_0}{(L + L')} \left( \lim_{z \to 0} \frac{e^{tz} (1 + A \ z)}{z(z + B)} z \right. \]

\[ + \left. \lim_{z \to -B} \frac{e^{tz} (1 + A \ z)}{z(z + B)} (z + B) \right) \]  \hspace{1cm} (58)

The first residue contributes

\[ \frac{\epsilon_0 B}{(R + R')} \] .

The second residue contributes

\[ \frac{\epsilon_0}{(L + L')} e^{-Bt} \left( A - \frac{1}{B} \right) = \epsilon_0 e^{-(R+R')t} \left( \frac{1}{R} - \frac{1}{R + R'} \right) \]  \hspace{1cm} (60)

\[ = \epsilon_0 e^{-(R+R')t} \left( \frac{(R + R') - R}{R(R + R')} \right) \]  \hspace{1cm} (61)

\[ = \epsilon_0 e^{-(R+R')t} \left( \frac{R'}{R(R + R')} \right) \]  \hspace{1cm} (62)

So summing the two residues we have in total

\[ i(t) = \frac{\epsilon_0}{(R + R')} + \frac{\epsilon_0 R'}{R(R + R')} e^{-\frac{R+R'}{L+L'}t} \]  \hspace{1cm} (63)

which is the desired expression.