Assignment 7:

Do all four problems. Each correct solution will be given 25 points, with points taken away for errors (or for absence of explanation) according to the severity of the error.

Problem 1:
Consider the function, defined over the range $-L/2 \leq x \leq L/2$, 

$$
\begin{align*}
  f(x) &= 1 \quad \text{for } x > 0 \\
  f(x) &= -1 \quad \text{for } x < 0 \\
  f(0) = f(\pm L/2) &= 1/2 \\
\end{align*}
$$

(The values at 0 and $\pm L/2$ are irrelevant.)

A) Find the coefficients $F_k$ of its Fourier series expansion

$$
  f(x) = \frac{1}{\sqrt{L}} \sum_{k \neq 0} F_k e^{2\pi i k x / L} 
$$

(Note that $F_0 = 0$.)

B) By using a technique similar to the one used to sum the series representation of the Green’s function in the notes on the Fourier transform, sum the series 2 with the coefficients you found in part A) and show that the sum does indeed reproduce the given function.
Solution

A) The Fourier coefficients are given by

\[ F_k = \frac{1}{\sqrt{L}} \int_{-L/2}^{L/2} f(x) e^{-2\pi i k x/L} \, dx = \]

\[ \frac{1}{\sqrt{L}} \left[ \int_0^{L/2} e^{-2\pi i k x/L} \, dx - \int_{-L/2}^0 e^{-2\pi i k x/L} \, dx \right] = \]

\[ \frac{i \sqrt{L}}{2\pi k} \left[ e^{-\pi i k} - 1 - e^{\pi i k} \right] = \]

\[ \frac{i \sqrt{L}}{\pi k} [\cos(\pi k) - 1] \quad (3) \]

(Notice that \( F_{-k} = F_k^* \) as one would expect since \( f(x) \) is real.)

B) By inserting the Fourier coefficients we just found into Eq. 2 we get

\[ f(x) = \frac{1}{\pi} \sum_{k \neq 0} \frac{\cos(\pi k) - 1}{k} e^{2\pi i k x/L} \quad (4) \]

We write the sum as a contour integral over the path \( c_1 + c_2 \) used in Eq. 93, page 13, of the lecture notes on the Fourier transform:

\[ f(x) = \frac{i}{\pi} \int_{c_1+c_2} \frac{\pi}{z \sin z} (\cos(z) - 1) e^{2izx/L} (-e^{\pm iz}) \quad (5) \]

the last factor to remove the \((-1)^k\) factor from the residues at the poles. We want to close the contour with a circle at infinity. For the purpose we write the integral as the sum of three separate integrals, and we also absorb the negative sign in front of \( e^{\pm iz} \) by reversing the direction of the contour:

\[ f(x) = i \left[ \int_{-c_1-c_2} e^{iz} \frac{e^{2izx/L} (e^{\pm iz})}{z \sin z} \, dz + \right. \]

\[ \left. \int_{c_1+c_2} e^{-iz} \frac{e^{2izx/L} (e^{\pm iz})}{z \sin z} \, dz - 2 \int_{c_1+c_2} \frac{e^{2izx/L} (e^{\pm iz})}{z \sin z} \, dz \right] \quad (6) \]

In order to proceed we must pick up a sign for \( x \), so let us assume that \( x > 0 \). The function \( f(x) \) defined by Eq. 4 is, anyway, clearly antisymmetric, so we
only need to consider \( x > 0 \) or \( x < 0 \). In order to close the contour \(-c_1-c_2\) with a circle of radius \( R \), sending the \( R \to \infty \), we must make sure that the integrand vanishes fast enough for \( R \to \infty \). The factor \( \sin z \) at denominator gives a suppression factor \( e^{-R} \) both for \( z \to \infty \) and \( z \to -\infty \). To make sure that the numerators do not grow faster than \( e^{R} \), we choose, in \( e^{\pm iz} \), the negative sign in the first integral, the positive sign in the second, and, with \( x > 0 \) (this is where the choice \( x > 0 \) plays a role) the negative sign in the third integral. We can now now write

\[
f(x) = \frac{1}{2} \left[ \int_{\gamma} \frac{e^{2ixz/L}}{z \sin z} \, dz + \int_{\gamma} \frac{e^{2ixz/L}}{z \sin z} \, dz - 2 \int_{\gamma} \frac{e^{2ixz/L}e^{-iz}}{z \sin z} \, dz \right] = 2 \int_{\gamma} \frac{e^{2ixz/L}(1 - e^{-iz})}{z \sin z} \, dz = 2 \int_{\gamma} \frac{e^{2ixz/L}(iz + O(z^2))}{z \sin z} \, dz = -2 \int_{\gamma} \left[ \frac{e^{2ixz/L}}{ \sin z} + \text{analytic function} \right] \, dz \tag{7}
\]

where \( \gamma \) is the closed contour formed by \(-c_1 - c_2\) and the semicircles at infinity.

**Problem 2:**

The propagation of a wave in a dissipative medium is governed by the equation

\[
\frac{\partial^2 f(x, t)}{\partial t^2} = v^2 \frac{\partial^2 f(x, t)}{\partial x^2} + 2b \frac{\partial^3 f(x, t)}{\partial x^2 \partial t} \tag{8}
\]

The term with the third derivative induces a dissipation which increases as the wave-number \( p \) increases and the wave-length becomes smaller.

Write \( f(x, t) \) as a Fourier integral

\[
f(x, t) = \frac{1}{\sqrt{2\pi}} \int F(p, t) e^{ipx} \, dp \tag{9}
\]

A) Find \( F(p, t) \) assuming that

\[
F(p, 0) = F_0(p) \tag{10}
\]

\[
\left. \frac{\partial F(p, t)}{\partial t} \right|_{t=0} = 0 \tag{11}
\]

where \( F_0(p) \) is a given function. (It is the Fourier transform of the initial profile of the wave \( f(x, 0) \).)
B) As time evolves the function \( f(x, t) \) will decay to zero. Insert the solution you found for \( F(p, t) \) into the Fourier representation of the wave, Eq. 9. Write the integral as the sum of two separate integrals, one giving origin to an oscillatory damped behavior, the other to a damped behavior without oscillations.

**Solution**

In terms of the Fourier transform \( F(p) \) Eq. 8 becomes

\[
\frac{\partial^2 F(p, t)}{\partial t^2} + 2bp^2 \frac{\partial F(p, t)}{\partial t} + v^2 p^2 F(p, t) = 0
\]  

(12)

This is the equation for a damped harmonic oscillator. We look for solutions of the form

\[
F(p, t) = Ae^{i\omega t}
\]  

(13)

Substituting into Eq. 12 we obtain the equation for \( \omega \)

\[
-\omega^2 + 2ibp^2 \omega + v^2 p^2 = 0
\]  

(14)

with solution

\[
\omega_{\pm} = ibp^2 \pm p\sqrt{-b^2 p^2 + v^2}
\]  

(15)

It is convenient to define

\[
\Delta^2 = |v^2 - b^2 p^2|
\]  

(16)

Then, for \( p^2 \leq v^2 / b^2 \) the solution will be of the form

\[
F(p, t) = A_+ e^{[-bp^2 + ip\Delta(p)]t} + A_- e^{[-bp^2 - ip\Delta(p)]t}
\]  

(17)

while for \( p^2 > v^2 / b^2 \) it will be

\[
F(p, t) = A_+ e^{[-bp^2 + ip\Delta(p)]t} + A_- e^{[-bp^2 - ip\Delta(p)]t}
\]  

(18)

At \( t = 0 \), for \( p^2 \leq v^2 / b^2 \) we have

\[
F(p, 0) = A_+ + A_-
\]  

(19)

\[
\left. \frac{\partial F(p, t)}{\partial t} \right|_{t=0} = -bp^2 (A_+ + A_-) + ip\Delta(p)(A_+ - A_-)
\]  

(20)

and for \( p^2 > v^2 / b^2 \)

\[
F(p, 0) = A_+ + A_-
\]  

(21)

\[
\left. \frac{\partial F(p, t)}{\partial t} \right|_{t=0} = -bp^2 (A_+ + A_-) + p\Delta(p)(A_+ - A_-)
\]  

(22)
From the given initial conditions Eqs. 10, 11 we obtain, for $p^2 \leq v^2/b^2$

$$A_+ + A_- = F_0(p)$$  \hspace{1cm} (23)
$$A_+ - A_- = -i \frac{bpF_0(p)}{\Delta(p)}$$  \hspace{1cm} (24)

$$A_+ = \frac{F_0(p)}{2} \left[ 1 - i \frac{bp}{\Delta(p)} \right]$$  \hspace{1cm} (25)
$$A_- = \frac{F_0(p)}{2} \left[ 1 + i \frac{bp}{\Delta(p)} \right]$$  \hspace{1cm} (26)

$$F(p, t) = F_0(p)e^{-bp^2t} \left[ \cos p\Delta(p)t + \frac{bp}{\Delta(p)} \sin p\Delta(p)t \right]$$  \hspace{1cm} (27)

and for $p^2 > v^2/b^2$

$$A_+ + A_- = F_0(p)$$  \hspace{1cm} (28)
$$A_+ - A_- = \frac{bpF_0(p)}{\Delta(p)}$$  \hspace{1cm} (29)

$$A_+ = \frac{F_0(p)}{2} \left[ 1 + \frac{bp}{\Delta(p)} \right]$$  \hspace{1cm} (30)
$$A_- = \frac{F_0(p)}{2} \left[ 1 - \frac{bp}{\Delta(p)} \right]$$  \hspace{1cm} (31)

$$F(p, t) = F_0(p)e^{-bp^2t} \left[ \cosh p\Delta(p)t + \frac{bp}{\Delta(p)} \sinh p\Delta(p)t \right]$$  \hspace{1cm} (32)

We may now substitute the results obtained with Eqs. 27, 32 into the Fourier integral of Eq. 9 separating the two above ranges of $p$. Before doing so we notice that the r.h.s. in Eqs. 27, 32 are even in $p$, so we can combine the contributions from $F(-p) = F^*(p)$ and $F(p)$ into a single term and obtain our final result

$$f(x, t) = \sqrt{\frac{2}{\pi}} \left[ \int_0^{v/b} \text{Re} F_0(p)e^{-bp^2t} \left[ \cos p\Delta(p)t + \frac{bp}{\Delta(p)} \sin p\Delta(p)t \right] e^{ipx} \, dp ight.
+ \left. \int_{v/b}^{\infty} \text{Re} F_0(p)e^{-bp^2t} \left[ \cosh p\Delta(p)t + \frac{bp}{\Delta(p)} \sinh p\Delta(p)t \right] e^{ipx} \, dp \right]$$  \hspace{1cm} (33)
Equation 33 clearly shows the separation between a damped oscillatory component and a damped component with no oscillations.

**Problem 3:**

Consider the wave equation

\[ \frac{\partial^2 f(x, t)}{\partial t^2} - v^2 \frac{\partial^2 f(x, t)}{\partial x^2} = 0 \]  \hspace{1cm} (34)

If we look for a solution of the form

\[ f(x, t) = e^{-\omega t + ikx} \]  \hspace{1cm} (35)

where for definiteness we take \( \omega \) to be positive (\( k \) can be positive or negative,) inserting into the equation we find the constraint

\[ \omega = v|k| \]  \hspace{1cm} (36)

Equation 36 is called a dispersion formula. In computational applications, but also in the study of physical systems like a crystal, the continuum \( x \)-axis is replaced by a discrete set of points \( x_j \) with uniform spacing \( a \)

\[ x_j = ja \]  \hspace{1cm} (37)

A frequently used discretization of the second order partial derivative with respect to \( x \) is its “central difference approximation”

\[ \frac{\partial^2 f(x)}{\partial x^2} \approx \frac{f(x_{j+1}) + f(x_{j-1}) - 2f(x_j)}{a^2} \]  \hspace{1cm} (38)

With this the wave equation takes the form

\[ \frac{d^2 f_j(t)}{dt^2} - v^2 \frac{f_{j+1}(t) + f_{j-1}(t) - 2f_j(t)}{a^2} = 0 \]  \hspace{1cm} (39)

where we denoted \( f(x_j) \) by \( f_j \). Equation 39 still admits solutions of the form of Eq. 35, although \( k \) is now restricted to the range

\[ -\frac{\pi}{a} \leq k \leq \frac{\pi}{a} \]  \hspace{1cm} (40)

since \( k \) and \( k + 2\pi/a \) produce the same set of values \( f_j \).
A) Find the dispersion formula

$$\omega = \Omega(k)$$  \hspace{1cm} (41)

which follows from demanding that

$$f_j(t) = e^{-\omega t + ika}$$  \hspace{1cm} (42)

satisfies Eq. 39 and show that for small $k$, i.e. for long wave length, $\Omega(k)$ agrees with the continuum dispersion formula Eq. 36 with

$$\Omega(k) = v|k| + O(k^3)$$  \hspace{1cm} (43)

B) Find a discretization of the second derivative with respect to $x$

$$\frac{\partial^2 f(x)}{\partial x^2} \approx \frac{\alpha[f(x_{j+2}) + f(x_{j-2})] + \beta[f(x_{j+1}) + f(x_{j-1})] + \gamma f(x_j)}{a^2}$$  \hspace{1cm} (44)

which will origin to a dispersion formula that agrees with the continuum limit up to order $(ka)^3$ included, i.e. determine the coefficients $\alpha, \beta, \gamma$ in Eq. 44 so that

$$\Omega(k) = v|k| + O(k^5)$$  \hspace{1cm} (45)

Solution

A shift by $a$ in the variable $x$ of the function $e^{ikx}$ multiplies it by $e^{ika}$

$$e^{ik(x+a)} = e^{ika} e^{ikx}$$  \hspace{1cm} (46)

With this in mind we find that the action of the central difference approximation to the second derivative, which we denote by $D_2$, is

$$D_2 e^{ikx} = \frac{e^{ika} + e^{-ika} - 2}{a^2} e^{ikx}$$  \hspace{1cm} (47)

Assuming a solution of the form 35 and inserting into Eq. 39 we find the dispersion formula

$$-\omega^2 = v^2 \frac{e^{ika} + e^{-ika} - 2}{a^2} = -v^2 \frac{2 - 2 \cos ka}{a^2} = -v^2 \frac{4 \sin^2 ka/2}{a^2}$$  \hspace{1cm} (48)

or

$$\omega = 2v \left| \frac{\sin(ka/2)}{a} \right|$$  \hspace{1cm} (49)
Expanding \(\sin(ka/2)\) for small \(k\) we find

\[
\omega = v|k - k(ka)^2/24 + \ldots| = v|k| + O(k^3)
\]  

(50)

B) Let us denote by \(D_4\) the operators in the r.h.s. of Eq. 44. We have

\[
D_4 e^{ikx} = \frac{\alpha(e^{2ika} + e^{-2ika}) + \beta(e^{ika} + e^{-ika}) + \gamma}{a^2} e^{ikx}
\]  

(51)

Expanding for small \(k\) we obtain

\[
D_4 e^{ikx} = \left[\frac{2\alpha + 2\beta + \gamma}{a^2} - [4\alpha + \beta]k^2 + \left[\frac{4\alpha}{3} + \frac{\beta}{12}\right]k^4 + O(k^5)\right] e^{ikx}
\]  

(52)

This expression will agree with \(|k^2|\) up to order \(k^4\) included if

\[
2\alpha + 2\beta + \gamma = 0
\]  

(53)

\[
4\alpha + \beta = 1
\]  

(54)

\[
\frac{4\alpha}{3} + \frac{\beta}{12} = 0
\]  

(55)

These equations are solved by

\[
\alpha = -\frac{1}{12}
\]  

(56)

\[
\beta = \frac{4}{3}
\]  

(57)

\[
\gamma = \frac{5}{2}
\]  

(58)

With these parameters we have

\[
D_4 e^{ikx} = [k^2 + O(k^5)] e^{ikx}
\]  

(59)

and inserting into the wave equation, and taking the square root, we find the dispersion formula

\[
\omega = v|k| + O(k^5)
\]  

(60)

**Problem 4:**

Let the function \(f(x, t)\) satisfies the wave equation

\[
\frac{\partial^2 f(x, t)}{\partial t^2} - v^2 \frac{\partial^2 f(x, t)}{\partial x^2} = 0
\]  

(61)
Write $f(x, t)$ as a Fourier integral

$$f(x, t) = \frac{1}{\sqrt{2\pi}} \int F(p, t) e^{ipx} \quad (62)$$

Denote the value of $f(x, 0)$ by $f_0(x)$ and the value of $F(p, 0)$ by $F_0(p)$. We will of course have

$$f_0(x) = \frac{1}{\sqrt{2\pi}} \int F_0(p) e^{ipx} \quad (63)$$

A) Substitute the Fourier integral representation of $f(x, t)$ into the wave equation to find the equation satisfied by $F(p, t)$. Solve this equation with initial conditions

$$F(p, 0) = F_0(p) \quad (64)$$

$$\left. \frac{\partial F(p, t)}{\partial t} \right|_0 = -ivpF_0(p) \quad (65)$$

Insert your solution back into Eq. 62 and show that the result represents a wave traveling forward with velocity $v$.

B) Imagine that the velocity parameter $v$ in Eq. 61 is not constant and that the equation take the form

$$\frac{\partial^2 f(x, t)}{\partial t^2} - v^2[1 + \epsilon w(x)] \frac{\partial^2 f(x, t)}{\partial x^2} = 0 \quad (66)$$

where $\epsilon$ is a parameter which we will eventually take to be very small. Denote by $W(p)$ the Fourier transform of $w(x)$ so that

$$w(x) = \frac{1}{\sqrt{2\pi}} \int W(p) e^{ipx} \quad (67)$$

Use the convolution theorem to write the evolution equation now satisfied by $F(p, t)$ as an integro-differential equation, i.e. an equation that will involve the second time derivative of $F(p, t)$ as before, but also an integral involving $F$ and $W$.

C) Assume that $\epsilon$ is small enough to warrant looking for the solution as a perturbation of order $\epsilon$ of the solution with constant $v$. Thus look for a solution

$$F(p, t) + \epsilon G(p, t) \quad (68)$$
where $F(p, t)$ is the solution you found in part A. Substitute into the equation you found in part B), keeping only terms of order $\epsilon$. You should find that the time evolution of $G(p, t)$ obeys a harmonic oscillator equation with a driving term. In particular, take $F_0(p) = \delta(p - p_0)$, which corresponds to a monochromatic wave of wave-number $p_0$ and insert it into the equation for $G$. You will see that the variable velocity will excite a whole distribution of wave components with variable wave number. You do not need to proceed any further with the solution, which would require specifying the functions $F_0(p)$ and $W(p)$.

**Solution**

A) The second derivative operator acts in Fourier space by $F(p) \rightarrow -p^2 F(p)$. Thus the equation satisfied by $F(p, t)$ is

$$\frac{\partial^2 F(p, t)}{\partial t^2} + v^2 p^2 F(p, t) = 0 \quad (69)$$

This equation has the two solutions

$$F(p, t) = e^{\pm ivpt} \quad (70)$$

with derivatives

$$\frac{\partial F(p, t)}{\partial t} = \pm ivp e^{\pm ivpt} \quad (71)$$

The initial conditions single out the solution with the negative sign in the exponent, so we have

$$F(p, t) = F_0(p) e^{-ivpt} \quad (72)$$

Inserting into Eq. 62 we find

$$f(x, t) = \frac{1}{\sqrt{2\pi}} \int F_0(p) e^{ip(x-vt)} = f_0(x - vt) \quad (73)$$

which is indeed the equation of a wave that travels forward with velocity $v$.

B) The Fourier transform of $\partial^2 f(x, t)/\partial^2 x$ is $-p^2 F(p, t)$ and the Fourier transform of $v^2 \epsilon w(x)$ is $v^2 \epsilon W(p)$. By the convolution theorem the Fourier transform of the product $v^2 [\epsilon W(p)][\partial^2 f(x, t)/\partial^2 x]$ is given by

$$-\frac{v^2 \epsilon}{\sqrt{\pi}} \int W(p - p') p'^2 F(p', t) dp' \quad (74)$$
With this, the Fourier transform of the whole equation becomes

\[ \frac{\partial^2 F(p, t)}{\partial t^2} + v^2 p^2 F(p, t) + \frac{v^2 \epsilon}{\sqrt{\pi}} \int W(p - p') p'^2 F(p', t) \, dp' = 0 \] (75)

which is an integro-differential equation for \( F(p, t) \).

C) Substituting the expression \( F_0(p) e^{-ivpt} + \epsilon G(p, t) \) for \( F(p, t) \) into Eq. 75 we find that the terms of order 1 cancel and that the terms of order \( \epsilon \) result in the equation

\[ \frac{\partial^2 G(p, t)}{\partial t^2} + v^2 p^2 G(p, t) + \frac{v^2}{\sqrt{\pi}} \int W(p - p') p'^2 F_0(p') e^{-ivp't} \, dp' = 0 \] (76)

or

\[ \frac{\partial^2 G(p, t)}{\partial t^2} + v^2 p^2 G(p, t) = R(p, t) \] (77)

where

\[ R(p, t) = -\frac{v^2}{\sqrt{\pi}} \int W(p - p') p'^2 F_0(p') e^{-ivp't} \, dp' \] (78)

is a known function of \( p \) and \( t \). (Or, at least, a function which can in principle be calculated since \( W(p) \) and \( F_0(p) \) are known.) Thus we see that \( G(p, t) \) satisfies a forced harmonic oscillator equation which could be solved with the method of variation of constants. If we take \( F_0(p) = \delta(p - p_0) \), then

\[ R(p, t) = -\frac{v^2}{\sqrt{\pi}} \int W(p - p') p'^2 \delta(p' - p_0) e^{-ivp't} \, dp' = W(p - p_0) p_0^2 e^{-ivp_0 t} \] (79)

and the equation for the perturbation \( G(p, t) \) becomes

\[ \frac{\partial^2 G(p, t)}{\partial t^2} + v^2 p^2 G(p, t) = W(p - p_0) p_0^2 e^{-ivp_0 t} \] (80)

We see from this equation that a whole range of wave-numbers \( p \) will be excited, depending on the function \( W(p) \), i.e. the Fourier transform of the variable velocity term \( w(x) \).