Assignment 6:

Do all four problems. Each correct solution will be given 25 points, with points taken away for errors (or for absence of explanation) according to the severity of the error.

Problem 1:

A uniform metal ring of length \( L \) has a temperature distribution \( f(x, t) \), with \( 0 \leq x \leq L \). The heat diffuses according to the equation

\[
\frac{\partial \phi(x, t)}{\partial t} = D \frac{\partial^2 \phi(x, t)}{\partial x^2}
\]

(1)

where \( D \) is the diffusion constant. Assume an initial heat distribution

\[
\phi(x, 0) = T, \quad \text{for } \frac{L - d}{2} \leq x \leq \frac{L + d}{2}
\]

(2)

and \( \phi(x, 0) = 0 \) outside of the above interval. In words: the initial temperature of the metal is \( T \) in a section of length \( d \) in the middle of the ring, and is zero in the rest of the ring.

Use the Fourier transform to express the temperature for \( t > 0 \) as a Fourier series. You should find that the temperature will asymptotically tend to a constant value \( T_\infty \) with all non-zero Fourier components decaying exponentially. Specify the value of \( T_\infty \) and the rate of decay \( \gamma = 1/\tau \) of the Fourier components with the slowest rate of decay. Calculate the values of the sum of the zero Fourier component and the components with the slowest rate of decay at \( t = \tau/2 \) for \( x = 0, L/8, L/4, 3L/8, L/2 \). Optional: instead of calculating those values you may plot the corresponding field profile.

Note: If you find it convenient, you are allowed to shift the \( x \) coordinate by \(-L/2\), which will make the \( x \)-range \(-L/2 \leq x \leq L/2\) and will place the initially heated region at \(-d/2 \leq x \leq d/2\).
Solution

We expand $\phi(x, t)$ into a Fourier series

$$
\phi(x, t) = \frac{1}{\sqrt{L}} \sum_k \Phi_k(t) e^{\frac{2\pi ikx}{L}} \tag{3}
$$

Inserting the expansion into Eq. 1 we obtain

$$
\frac{\partial \phi(x, t)}{\partial t} = \frac{1}{\sqrt{L}} \sum_k \frac{d\Phi_k(t)}{dt} e^{\frac{2\pi ikx}{L}} =
$$

$$
D \frac{\partial^2 \phi(x, t)}{\partial x^2} = D \frac{1}{\sqrt{L}} \sum_k \Phi_k(t) \frac{\partial^2}{\partial x^2} e^{\frac{2\pi ikx}{L}} =
$$

$$
-4\pi^2 D \frac{1}{L^2} \sum_k k^2 \Phi_k(t) e^{\frac{2\pi ikx}{L}} \tag{4}
$$

From this equation it follows that the evolution of the Fourier components is uncoupled, each one satisfying the equation

$$
\frac{d\Phi_k(t)}{dt} = -\frac{4D\pi^2 k^2}{L^2} \Phi_k(t) \tag{5}
$$

This equation is solved by

$$
\Phi_k(t) = e^{-\frac{4D\pi^2 k^2 t}{L^2}} \Phi_k(0) \tag{6}
$$

which gives

$$
\phi(x, t) = \frac{1}{\sqrt{L}} \sum_k e^{\frac{2\pi ikx}{L}} e^{-\frac{4D\pi^2 k^2 t}{L^2}} \Phi_k(0) \tag{7}
$$

We still need to calculate $\Phi_k(0)$. Using the range $-L/2 \leq x \leq L/2$, the Fourier components $\Phi_k(0)$ are given by

$$
\Phi_k(0) = \frac{1}{\sqrt{L}} \int_{-d/2}^{d/2} T e^{-\frac{2\pi ikx}{L}} dx = \frac{dT \sqrt{L}}{2\pi k} \left[ e^{-\frac{\pi kd}{L}} - e^{\frac{\pi kd}{L}} \right] = \frac{T \sqrt{L}}{\pi k} \sin \frac{kd}{L} \tag{8}
$$

for $k \neq 0$ and

$$
\Phi_0(0) = \frac{1}{\sqrt{L}} \int_{-d/2}^{d/2} T dx = \frac{dT}{\sqrt{L}} \tag{9}
$$
Inserting these results into Eq. 7 we obtain

$$\phi(x, t) = \frac{dT}{L} + \frac{1}{\pi} \sum_{k \neq 0} e^{\frac{2\pi k x}{L}} e^{-\frac{4D\pi^2 k^2 t}{L^2}} \sin \frac{kd}{L} =$$

$$= \frac{dT}{L} + \frac{1}{\pi} \sum_{k > 0} \left[ e^{\frac{2\pi k x}{L}} + e^{-\frac{2\pi k x}{L}} \right] e^{-\frac{4D\pi^2 k^2 t}{L^2}} \sin \frac{kd}{L} =$$

$$\frac{dT}{L} + \frac{2}{\pi} \sum_{k > 0} \frac{1}{k} \cos \frac{2\pi k x}{L} \sin \frac{kd}{L} \exp \left(-\frac{4D\pi^2 k^2 t}{L^2}\right)$$

(10)

From this expression we see that all the Fourier components of $\phi(x, t)$ with non-zero index will decay to zero, so that the temperature distribution will tend to $dT/L$, which makes sense because it shows that the heat will distribute evenly though the whole ring. Insofar as the rate of decay of the Fourier components, the one with the slowest rate of decay is the one with index $k = 1$ in the sum of Eq. 10. Limiting the sum to the $k = 0$ and $k = 1$ terms we get

$$\phi(x, t) \sim \frac{dT}{L} + \frac{2}{\pi} \cos \frac{2\pi x}{L} \sin \frac{d}{L} \exp \left(-\frac{4D\pi^2 t}{L^2}\right)$$

(11)
The rate of decay of the $k = 1$ component is

$$\gamma = \frac{4D\pi^2}{L^2} \quad (12)$$

At $t = \tau = 1/\gamma$ Eq. 11 gives

$$\phi(x, t) \sim \frac{dT}{L} + \frac{2}{\pi} \cos \frac{2\pi x}{L} \sin \frac{d}{L} e^{-1} \quad (13)$$

This function with $L = 10$, $d = 2$, $T = 4$ is shown in Figure 1.

**Problem 2:**

Starting from the Fourier series for a complex function $f(x)$ defined over the interval $0 \leq x \leq L$:

$$f(x) = \frac{1}{\sqrt{L}} \sum_k F_k e^{\frac{2\pi ikx}{L}} \quad (14)$$

develop the Fourier sine series expansion for a real function $g(x)$ defined over the interval $0 \leq x \leq d$ with Dirichlet boundary conditions $f(0) = f(d) = 0$.

**Hint:** Extend the definition of $g(x)$ to a function $f(x)$ defined over $-d \leq x \leq d$ with $f(x) = g(x)$ for $0 \leq x \leq d$ and a suitable definition of $f(x)$ in the rest of the interval to make it satisfy periodic boundary conditions.

**Solution**

We define

$$f(x) = g(x) \quad \text{for } 0 \leq x \leq d \quad (15)$$

$$f(x) = -g(-x) \quad \text{for } -d \leq x \leq 0 \quad (16)$$

With this definition $f(-d) = f(d) = 0$, $f'(-d) = f'(d) = g'(d)$, guaranteeing that $f$ satisfies periodic boundary conditions. Also, with obvious notation $f(0-) = f(0+) = 0$, $f'(0-) = f'(0+) = g'(0)$, guaranteeing continuity of $f$ and its derivative at $x = 0$. From the equation for the inverse Fourier transform

$$F_k = \frac{1}{\sqrt{2d}} \int_{-d}^d f(x) e^{-\frac{2\pi ikx}{2d}} \, dx \quad (17)$$
we get

\[ F_k = \frac{1}{\sqrt{2d}} \left[ \int_0^d f(x) e^{-\frac{2\pi ikx}{2d}} \, dx + \int_{-d}^0 f(x) e^{-\frac{2\pi ikx}{2d}} \, dx \right] = \]

\[ \frac{1}{\sqrt{2d}} \left[ \int_0^d g(x) e^{-\frac{2\pi ikx}{2d}} \, dx - \int_{-d}^0 g(-x) e^{\frac{2\pi ikx}{2d}} \, dx \right] = \]

\[ \frac{1}{\sqrt{2d}} \left[ \int_0^d g(x) e^{-\frac{2\pi ikx}{2d}} \, dx - \int_0^d g(x) e^{\frac{2\pi ikx}{2d}} \, dx \right] = \]

\[ \frac{1}{\sqrt{2d}} \int_0^d g(x) \left[ e^{-\frac{2\pi ikx}{2d}} - e^{\frac{2\pi ikx}{2d}} \right] \, dx = -\frac{2i}{\sqrt{2d}} \int_0^d g(x) \left[ \sin \frac{\pi kx}{d} \right] \, dx \quad (18) \]

Summarizing, we find

\[ F_k = \frac{-2i}{\sqrt{2d}} \int_0^d g(x) \left[ \sin \frac{\pi kx}{d} \right] \, dx \quad (19) \]

Notice that \( F_0 = 0 \). and for \( k \neq 0 \), \( F_{-k} = F_k^* \) as it should be for real \( f(x) \).

We insert now this result into Eq. 14 of the assignment to obtain

\[ f(x) = \frac{1}{\sqrt{L}} \sum_{k>0} F_k e^{\frac{2\pi ikx}{L}} + \sum_{k<0} F_k e^{\frac{2\pi ikx}{L}} = \]

\[ f(x) = \frac{1}{\sqrt{2d}} \left[ \sum_{k>0} F_k e^{\frac{\pi kx}{2d}} + \sum_{k<0} F_k e^{\frac{-\pi kx}{2d}} \right] \quad (20) \]

Using \( F_{-k} = F_k^* = -F_k \), since \( F_k \) is imaginary, see Eq. 19, this can be rewritten as

\[ f(x) = \frac{1}{\sqrt{2d}} \sum_{k>0} F_k \left[ e^{\frac{\pi kx}{d}} - e^{-\frac{\pi kx}{d}} \right] = \frac{2i}{\sqrt{2d}} \sum_{k>0} F_k \sin \frac{\pi kx}{d} \quad (21) \]

It is convenient to define

\[ F_k = -iG_k \quad (22) \]

With this, Eqs. 19 and 21 take the form

\[ G_k = \frac{\sqrt{2}}{d} \int_0^d g(x) \left[ \sin \frac{\pi kx}{d} \right] \, dx \quad (23) \]
and
\[ g(x) = \sqrt{\frac{2}{d}} \sum_{k>0} G_k \sin \frac{\pi k x}{d} \]  \hspace{1cm} (24)

which express the full content of the Fourier sine series. Note that all the variables in these two equations are real.

One may observe that the sequence of functions
\[ f^{(k)}(x) = \sqrt{\frac{2}{d}} \sin \frac{\pi k x}{d} \]  \hspace{1cm} (25)

for an orthonormal set since
\[ \int_0^d f^{(k)}(x) f^{(k')} (x) \, dx = \frac{2}{d} \sin \frac{\pi k x}{d} \sin \frac{\pi k' x}{d} \, dx = \delta_{k,k'} \]  \hspace{1cm} (26)

and, assuming the expansion in Eq. 24, derive then Eq. 23. But this would not, by itself, establish the Fourier sine series transform, because one would still need to prove the completeness of the expansion in Eq. 24, which cannot be assumed. Indeed, the sequence of functions formed by the functions \( f^{(k)}(x) \) with \( k \) odd also forms an orthonormal set, but the expansion
\[ f(x) = \sqrt{\frac{2}{d}} \sum_{k>0, \text{odd}} G_k \sin \frac{\pi k x}{d} \]  \hspace{1cm} (27)

could not represent an arbitrary function \( f(x) \) satisfying the boundary conditions \( f(0) = f(d) = 0 \). The advantage of starting from the Fourier series in Eq. 14 is that the completeness of that series has been established by the procedure of starting from a finite Fourier transform and going then to the continuum limit.

**Problem 3:**

Following a procedure similar to the one for problem 2, develop the Fourier cosine series expansion for a real function \( g(x) \) defined over the interval \( 0 \leq x \leq d \) with Neumann boundary conditions \( f'(0) = f'(d) = 0 \).

**Solution**

We define
\[ f(x) = g(x) \quad \text{for} \quad 0 \leq x \leq d \]  \hspace{1cm} (28)
\[ f(x) = g(-x) \quad \text{for} \quad -d \leq x \leq 0 \]  \hspace{1cm} (29)
With this definition \( f(-d) = f(d) \), \( f'(-d) = -f'(d) = -g'(d) = 0 \), guaranteeing that \( f \) satisfies periodic boundary conditions. Also, with obvious notation \( f(0-) = f(0+) \), \( f'(0-) = -f'(0+) = -g'(0) = 0 \), guaranteeing continuity of \( f \) and its derivative at \( x = 0 \). From the equation for the inverse Fourier transform

\[
F_k = \frac{1}{\sqrt{2d}} \int_{-d}^{d} f(x) e^{-\frac{2\pi ikx}{2d}} \, dx
\]  

we get

\[
F_k = \frac{1}{\sqrt{2d}} \left[ \int_{0}^{d} f(x) e^{-\frac{2\pi ikx}{2d}} \, dx + \int_{-d}^{0} f(x) e^{-\frac{2\pi ikx}{2d}} \, dx \right] =
\]

\[
= \frac{1}{\sqrt{2d}} \left[ \int_{0}^{d} g(x) e^{-\frac{2\pi ikx}{2d}} \, dx + \int_{-d}^{0} g(-x) e^{-\frac{2\pi ikx}{2d}} \, dx \right] =
\]

\[
= \frac{1}{\sqrt{2d}} \left[ \int_{0}^{d} g(x) e^{-\frac{2\pi ikx}{2d}} \, dx + \int_{0}^{d} g(x) e^{\frac{2\pi ikx}{2d}} \, dx \right] =
\]

\[
= \frac{1}{\sqrt{2d}} \int_{0}^{d} g(x) \left[ e^{-\frac{2\pi ikx}{2d}} + e^{\frac{2\pi ikx}{2d}} \right] \, dx = \frac{2}{\sqrt{2d}} \int_{0}^{d} g(x) \left[ \cos \frac{\pi kx}{d} \right] \, dx
\]  

Summarizing, we find

\[
F_k = \frac{2}{\sqrt{2d}} \int_{0}^{d} g(x) \left[ \cos \frac{\pi kx}{d} \right] \, dx
\]  

Notice that \( F_0 \) is real and, for \( k \neq 0 \), \( F_{-k} = F_k^* \) as it should be for real \( f(x) \).

We insert now this result into Eq. 14 of the assignment to obtain

\[
f(x) = \frac{1}{\sqrt{L}} \left[ F_0 + \sum_{k>0} F_k e^{\frac{\pi ikx}{d}} + \sum_{k<0} F_k e^{\frac{-\pi ikx}{d}} \right] =
\]

\[
f(x) = \frac{1}{\sqrt{2d}} \left[ F_0 + \sum_{k>0} F_k e^{\frac{\pi ikx}{d}} + \sum_{k<0} F_k e^{\frac{-\pi ikx}{d}} \right]
\]  

Using \( F_{-k} = F_k^* = F_k \), since \( F_k \) is real, see Eq. 32, this can be rewritten as

\[
f(x) = \frac{1}{\sqrt{2d}} \left[ F_0 + \sum_{k>0} F_k \left( e^{\frac{\pi ikx}{d}} - e^{\frac{-\pi ikx}{d}} \right) \right]
\]

\[
= \frac{1}{\sqrt{2d}} F_0 + \frac{2}{\sqrt{2d}} \left[ \sum_{k>0} F_k \cos \frac{\pi kx}{d} \right]
\]
It is convenient to define
\[ F_k = G_k \]  \hspace{1cm} (35)

With this, Eqs. 32 and 34 take the form
\[ G_k = \sqrt{\frac{2}{d}} \int_0^d g(x) \left[ \cos \frac{\pi k x}{d} \right] dx \]  \hspace{1cm} (36)

and
\[ g(x) = \sqrt{\frac{1}{2d}} G_0 + \sqrt{\frac{2}{d}} \sum_{k>0} G_k \cos \frac{\pi k x}{d} \]  \hspace{1cm} (37)

which express the full content of the Fourier cosine series.

**Problem 4:**
The time dependent wave function \( \psi(x, t) \) in one dimension evolves in time according to the Schrödinger equation
\[ i\hbar \frac{\partial \psi(x, t)}{\partial t} = -\frac{\hbar^2}{2m} \frac{\partial^2 \psi(x, t)}{\partial x^2} + V(x) \psi(x, t) \]  \hspace{1cm} (38)

We consider here the case of a free particle with \( V(x) = 0 \), so that the equation reduces to
\[ i\hbar \frac{\partial \psi(x, t)}{\partial t} = -\frac{\hbar^2}{2m} \frac{\partial^2 \psi(x, t)}{\partial x^2} \]  \hspace{1cm} (39)

Take for the initial value of the wave function a Gaussian wave packet with
\[ \psi(x, 0) = \frac{1}{4\sqrt{\pi} \sqrt{w}} \exp \left( -\frac{x^2}{2w^2} + \frac{i p_0 x}{\hbar} \right) \]  \hspace{1cm} (40)

Note that \( \hbar/w \) has dimension of momentum. In the relatively complex algebra needed to solve this problem it will be convenient to parametrize the Gaussian wave packet in terms of \( b = \hbar/w \) rather than \( w \), so we will write the initial value of \( \psi \) as
\[ \psi(x, 0) = \frac{1}{4\sqrt{\pi} \sqrt{b}} \exp \left( -\frac{b^2 x^2}{2\hbar^2} + \frac{i p_0 x}{\hbar} \right) \]  \hspace{1cm} (41)
As you will see, the above wave packet has a Gaussian shape also in momentum space, i.e. the space of its Fourier components, and \( b \) turns out to be its width in momentum space.

Note: The wave function in Eq. 41 is normalized to one: \( \int_{-\infty}^{\infty} |\psi(x, 0)|^2 \, dx = 1 \), and this normalization is preserved in the time evolution. In order to simplify the algebra, it is convenient to work with the non-normalized wave-function

\[
\psi(x, 0) = \exp \left( - \frac{b^2 x^2}{2\hbar^2} + \frac{ip_0x}{\hbar} \right) \tag{42}
\]

multiplying the final result, if necessary, by the normalization factor

\[
N = \frac{1}{\sqrt{4\pi b\hbar}} \tag{43}
\]

A) Use a Fourier transformation

\[
\psi(x, t) = \frac{1}{\sqrt{2\pi\hbar}} \int_{-\infty}^{\infty} \phi(p, t) e^{iapx/\hbar} \, dp \tag{44}
\]

to solve for the time evolution of the wave packet, i.e. to find the expression of \( \psi(x, t) \).

Note: This problem requires careful algebra, so, as a check, the solution should come out to be

\[
\psi(x, t) = N \times \left( 1 + i\frac{b^2 t}{2m\hbar} \right)^{-1/2} \times \exp \left( - \frac{[bx/\hbar - bp_0t/(m\hbar)]^2}{2[1 + ib^2t/(m\hbar)]} + ip_0x/\hbar - i\frac{p_0^2 t}{2m\hbar} \right) \tag{45}
\]

where \( N \) is the normalization factor given above.

B) The momentum operator in quantum mechanics is

\[
\hat{\hat{p}} = -i\hbar \frac{d}{dx} \tag{46}
\]

and its expectation value is

\[
\langle \hat{\hat{p}} \rangle = -i\hbar \int_{-\infty}^{\infty} \psi^*(x) \frac{d\psi(x)}{dx} \, dx \tag{47}
\]
(This equation assumes that $\psi(x)$ is normalized to one: $\int_{-\infty}^{\infty} |\psi(x,0)|^2 \, dx = 1$. Otherwise the r.h.s. should be divided by $\int_{-\infty}^{\infty} |\psi(x,0)|^2 \, dx$.)

Calculate the expectation value of $\hat{p}$ for the wave function $\psi(x,t)$ you found in part A) and show that it is constant in time.

 Hint: It is much easier to calculate $\langle \hat{p} \rangle$ in Fourier space.

**Solution**

A) Following the suggestion in the problem we dispose of the normalization and take

$$\psi(x,0) = \exp \left( -\frac{b^2 x^2}{2\hbar^2} + \frac{i p_0 x}{\hbar} \right)$$  \hspace{1cm} (48)

as initial value of the wave function. Its Fourier components are

$$\phi(p,0) = \frac{1}{\sqrt{2\pi\hbar}} \int_{-\infty}^{\infty} \exp(-ipx/\hbar) \exp \left( -\frac{b^2 x^2}{2\hbar^2} + \frac{ipx}{\hbar} \right) dx =$$

$$\frac{1}{\sqrt{2\pi\hbar}} \int_{-\infty}^{\infty} \exp \left( -\frac{b^2 x^2}{2\hbar^2} + \frac{-i(p-p_0)x}{\hbar} \right) dx \hspace{1cm} (49)$$

Gaussian integrals like this one are done by completing the square:

$$\phi(p,0) = \frac{1}{\sqrt{2\pi\hbar}} \int_{-\infty}^{\infty} \exp \left( -\frac{b^2 x^2}{2\hbar^2} - \frac{i(p-p_0)x}{\hbar} + \frac{(p-p_0)^2}{2} - \frac{(p-p_0)^2}{2} \right) dx =$$

$$\frac{1}{\sqrt{2\pi\hbar}} \int_{-\infty}^{\infty} \exp \left( -\frac{(p-p_0)^2}{2} \right) \int_{-\infty}^{\infty} \exp \left[ \frac{bx/\hbar + i(p-p_0)^2}{2} \right] dx =$$

$$\frac{1}{\sqrt{2\pi\hbar}} \exp \left( -\frac{(p-p_0)^2}{2} \right) \frac{\hbar\sqrt{2}}{b} \int_{-\infty}^{\infty} \exp(-y^2) dy =$$

$$\frac{\sqrt{\hbar}}{b} \exp \left( -\frac{(p-p_0)^2}{2b^2} \right) \hspace{1cm} \text{(with} \hspace{1cm} \frac{bx/\hbar + i(p-p_0)}{\sqrt{2}} = y) \hspace{1cm} (50)$$

From $\hat{p} = -i\hbar \frac{d}{dx}$ it follows that in Fourier space

$$\hat{p} \phi(p,t) = p \phi(p,t)$$  \hspace{1cm} (51)

Thus the Schrödinger equation 39 becomes

$$i\hbar \frac{\partial \phi(p,t)}{\partial t} = \frac{p^2}{2m} \phi(p,t)$$  \hspace{1cm} (52)
This equation is solved by

\[
\phi(p, t) = \phi(p, 0) \exp \left( - \frac{ip^2t}{2m\hbar} \right)
\]  

(53)

Inserting the initial value of \( \phi \) we obtain

\[
\phi(p, t) = \frac{\sqrt{\hbar}}{b} \exp \left( - \frac{(p - p_0)^2}{2b^2} - \frac{ip^2t}{2m\hbar} \right)
\]  

(54)

We insert this result into the expression for \( \psi \) in terms of \( \phi \), Eq. 44 to obtain

\[
\psi(x, t) = \frac{\sqrt{\hbar}}{\sqrt{2\pi b}} \int_{-\infty}^{\infty} \exp \left( - \frac{(p - p_0)^2}{2b^2} - i \frac{p^2t}{2m\hbar} + \frac{ip_0x}{\hbar} \right) dp
\]  

(55)

This is again a Gaussian integral which we can do by completing the square. It is convenient, however, to change variable first to \( q = p - p_0 \). With this we obtain, first,

\[
\psi(x, t) = \frac{1}{\sqrt{2\pi b}} \int_{-\infty}^{\infty} \exp \left( - \frac{q^2}{2b^2} - i \frac{(q + p_0)^2t}{2m\hbar} + i \frac{(q + p_0)x}{\hbar} \right) dq =
\]

\[
\frac{1}{\sqrt{2\pi b}} \int_{-\infty}^{\infty} \exp \left( - \frac{q^2}{2b^2} - i \frac{q^2t}{2m\hbar} - i \frac{q_0t}{m\hbar} + i \frac{qx}{\hbar} dq + i \frac{p_0x}{\hbar} \right) dq =
\]

\[
\frac{1}{\sqrt{2\pi b}} \int_{-\infty}^{\infty} \exp \left( - \frac{q^2}{2b^2} \left( 1 + i \frac{b^2t}{m\hbar} \right) + i \frac{q}{\hbar} \left( x - \frac{p_0t}{m} \right) + i \frac{p_0x}{\hbar} - i \frac{p_0^2t}{2m\hbar} \right) dq
\]  

(56)

As a further small simplification we set \( q/b = s \), obtaining

\[
\psi(x, t) =
\]

\[
\frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} \exp \left( - \frac{s^2}{2} \left( 1 + i \frac{b^2t}{m\hbar} \right) + i \frac{bs}{\hbar} \left( x - \frac{p_0t}{m} \right) + i \frac{p_0x}{\hbar} - i \frac{p_0^2t}{2m\hbar} \right) ds
\]  

(57)
We may now complete the square, getting

\[
\psi(x, t) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} \exp \left( -\frac{1}{2} \left[ s \sqrt{1 + \frac{b^2 t}{\hbar}} \frac{-b}{\hbar} \sqrt{1 + \frac{ib^2 t}{(2m\hbar)}} \right]^2 \right) \]

\[
- \frac{[bx/\hbar - b p_0 t/(m\hbar)]^2}{2[1 + ib^2 t/(m\hbar)]} + \frac{p_0 x}{\hbar} - \frac{i p_0^2 t}{2m\hbar} \right) ds =
\]

\[
\frac{1}{\sqrt{2\pi}} \exp \left( -\frac{[bx/\hbar - b p_0 t/(m\hbar)]^2}{2[1 + ib^2 t/(m\hbar)]} + \frac{p_0 x}{\hbar} - \frac{i p_0^2 t}{2m\hbar} \right)
\times \int_{-\infty}^{\infty} \exp \left( -\frac{[bx/\hbar - b p_0 t/(m\hbar)]^2}{2[1 + ib^2 t/(m\hbar)]} + \frac{p_0 x}{\hbar} - \frac{i p_0^2 t}{2m\hbar} \right) ds =
\]

\[
(1 + \frac{b^2 t}{2m\hbar})^{-1/2} \exp \left( -\frac{[bx/\hbar - b p_0 t/(m\hbar)]^2}{2[1 + ib^2 t/(m\hbar)]} + \frac{p_0 x}{\hbar} - \frac{i p_0^2 t}{2m\hbar} \right)
\] (58)

or, if we want the normalized wave function

\[
\psi(x, t) = \frac{1}{\sqrt{2\pi}} \sqrt{\frac{\hbar}{b}} \left(1 + \frac{b^2 t}{2m\hbar}\right)^{-1/2} \times \exp \left( -\frac{[bx/\hbar - b p_0 t/(m\hbar)]^2}{2[1 + ib^2 t/(m\hbar)]} + \frac{p_0 x}{\hbar} - \frac{i p_0^2 t}{2m\hbar} \right)
\] (59)

B) The momentum operator is diagonal in momentum space, i.e. it acts on the momentum components \( \phi(p, t) \) by

\[
\hat{p} \phi(p, t) = p \phi(p, t)
\] (60)

Its expectation value is therefore given by

\[
\langle \hat{p} \rangle = \int_{-\infty}^{\infty} \phi^* (p, t) p \phi(p, t) \, dp = \int_{-\infty}^{\infty} p |\phi(p, t)|^2 \, dp
\] (61)

assuming that the wave function is normalized. Now from the solution for \( \phi(p, t) \) found in part A, namely

\[
\phi(p, t) = N \sqrt{\frac{\hbar}{b}} \exp \left( -\frac{(p - p_0)^2}{2b^2} - \frac{ip^2 t}{2m\hbar} \right)
\] (62)

it is clear that

\[
|\phi(p, t)|^* = N^2 \sqrt{\frac{\hbar}{b}} \exp \left( -\frac{(p - p_0)^2}{b^2} \right)
\] (63)
is constant in time and has the shape of a Gaussian centered at $p = p_0$. We conclude that the expectation value of the momentum operator is $p_0$ and it is, indeed, constant in time.