10/28/19 - Lecture

Derivation of Fourier series from Discrete Transform

Let there be some function \( f(x) \) which is discrete on the interval \(-L/2 \leq x \leq L/2\). This interval can be divided into \( N \) smaller intervals of length \( a = \frac{L}{N} \). Our function \( f_j = f(x_j) \), where \( x_j = ja \) for \( j = -\frac{N}{2}, ..., \frac{N}{2} - 1 \). Taking the Fourier transform of \( f \) gives

\[
F_k = \sqrt{\frac{a}{N}} \sum_j f_j e^{-\frac{2\pi ik}{N}} = \sqrt{\frac{1}{L}} \sum_j a f_j e^{-\frac{2\pi i j k}{N}}, \quad \text{where} \quad \sqrt{\frac{1}{L}} = \sqrt{aN}. \tag{1}
\]

A sum can be written as an integral in the form

\[
\sum_j f_j \Delta x = \int_{-L/2}^{L/2} f(x) dx, \quad \text{where} \quad \Delta x \to 0. \tag{2}
\]

This applies to our case because as \( N \to \infty, a \to 0 \), allowing us to use it as our \( \Delta x \). Now we may translate between the discrete \( F_k \) and our function \( f(x) \) through

\[
F_k = \sqrt{\frac{1}{L}} \int_{-L/2}^{L/2} f(x) e^{-\frac{2\pi ikx}{L}} dx \tag{3}
\]

and

\[
f(x) = \sqrt{\frac{1}{L}} \sum_k F_k e^{\frac{2\pi ikx}{L}} dx. \tag{4}
\]

This is the transform over a finite range, so

\[
\sum_k |F_k|^2 = a \sum_j |f_j|^2 = \int_{-L/2}^{L/2} |f(x)|^2 dx \tag{5}
\]

will hold in these cases. In our function \( f(x) \) is periodic, \( f(x + mL) = f(x) \), where \( m \) is some integer, our series transformation holds outside of our interval, assuming the interval contains one full period of \( f(s) \). Note that we can also define infinitesimal shifts:

\[
f'(x) = \frac{df(x)}{dx}, \quad F_k' = \frac{2\pi ik}{L} F_k, \quad f(x) \to \tilde{f}(x) = f(x + b), \quad F_k \to \tilde{F}_k = e^{\frac{2\pi ikb}{L}} F_k.
\]

and furthermore if \( f(x) \) is real then; \( f^*(x) = \sqrt{\frac{1}{L}} \sum_k F^*_k e^{-\frac{2\pi ikx}{L}} = f(x) \implies F^*_k = F_{-k} \)
Completing the Square for Gaussian Integrals

If we are given some Gaussian integral of the form

$$ I = \int e^{-ax^2 + bx + c} dx = e^c \int e^{-ax^2 + bx} dx, $$

we must complete the square to evaluate. To complete the square we must add and subtract a $\frac{b^2}{4a}$ term from the exponent giving us the following integral

$$ I = e^c \int e^{-ax^2 - \frac{b^2}{4a} + \frac{b^2}{4a} + bx} dx = e^{(c + \frac{b^2}{4a})} \int e^{-a(x - \frac{b}{2a})^2} dx. $$

This integral can be evaluated with greater ease.

10/30/19 - Lecture

Dirac-Delta Function

Starting with the integral for $F(p)$ from $f(y)$,

$$ F(p) = \frac{1}{\sqrt{2\pi}} \int f(y) e^{-ipy} dy, $$

we can find $f(x)$ with the inverse transformation

$$ f(x) = \frac{1}{\sqrt{2\pi}} \int \left[ \frac{1}{\sqrt{2\pi}} \int f(y) e^{-ipy} dy \right] e^{ipx} dp. $$

This equation can be rewritten as

$$ f(x) = \frac{1}{2\pi} \int \int e^{ip(x-y)} df(y) dy = f(x) = \int f(y) \delta(x-y) dy, \text{ where } \delta(x-y) \equiv \frac{1}{2\pi} \int_{-\infty}^{\infty} e^{ip(x-y)} dp. $$

The Dirac-Delta function acts as a mapping:

$$ \frac{1}{L} \sum_{k=-\infty}^{\infty} e^{\frac{2\pi ik(x-y)}{L}} = \sum_{m=-\infty}^{\infty} \delta(x-y-mL), $$

$$ \frac{1}{L} \int_{-L/2}^{L/2} e^{\frac{2\pi ikx}{L}} dx = \delta_{k,k'}, $$

$$ \frac{1}{N} \sum_{k=0}^{N-1} e^{\frac{2\pi i(j'-mN)}{N}} dx = \delta_{j,(j'+mN)}. $$

Green’s Function

Let us take a differential equation of form

$$ -\frac{d^2 f(x)}{dx^2} = b(x), $$

on the interval $0 \leq x \leq L$ with Dirichlet boundary conditions $f(0) = f(L) = 0$. We will solve an analogous equation using the Green’s function

$$ -\frac{\partial^2 G(x,y)}{\partial x^2} = \delta(x-y). $$
We will now write our function \( f(x) \) as
\[
    f(x) = \int_{0}^{L} G(x, y)b(y)dy. \tag{16}
\]

Our differential equation can be written as
\[
    -\frac{d^2}{dx^2}f(x) = \int_{0}^{L} -\frac{d^2}{dx^2}G(x, y)b(y)dy = \int_{0}^{L} \delta(x - y)b(y)dy = b(x). \tag{17}
\]

We must now find Green’s function. \( G(x, y) \) must be zero on the boundaries, giving us
\[
    G(x, y) = \begin{cases} 
\alpha x, & \text{for } x < y \\
\beta(\frac{L}{y} - x), & \text{for } x > y 
\end{cases}. \tag{18}
\]

At the point \( x = y \), \( \alpha y = \beta(L - y) \), which results in \( \alpha = \frac{\beta}{y}L - \beta \). We may now write Green’s function as
\[
    G(x, y) = \begin{cases} 
\frac{\beta}{y}Lx - \beta x, & \text{for } x < y \\
\beta(L - x), & \text{for } x > y 
\end{cases}. \tag{19}
\]

Taking the derivative with respect to \( x \) of this function results in
\[
    \frac{\partial G(x, y)}{\partial x} = \begin{cases} 
\frac{\beta}{y}L - \beta, & \text{for } x < y \\
-\beta, & \text{for } x > y 
\end{cases}. \tag{20}
\]

The Dirac-Delta distribution described in Equation 15 must be normalized, implying that
\[
    \int_{0}^{L} \frac{\partial^2 G(x, y)}{\partial x^2} = 1. \tag{21}
\]

In order for this to be true, the discontinuity of \( \frac{\partial G}{\partial x} \) at \( y \) must be 1. From Equation 20, we can see that
\[
    \Delta \frac{\partial G}{\partial x} = \frac{\beta}{y}L - \beta - (-\beta) = \frac{\beta}{y}L. \tag{22}
\]

We can see that for \( \Delta \frac{\partial G}{\partial x} = 1 \), \( \beta = \frac{L}{y} \). We may now write out \( G(x, y) \) as
\[
    G(x, y) = \begin{cases} 
\frac{\pi(L - y)}{L}, & \text{for } x < y \\
\frac{(L - x)y}{L}, & \text{for } x > y 
\end{cases}. \tag{23}
\]

The differential equation
\[
    -\frac{d^2}{dx^2}f(x) = b(x), \tag{24}
\]
on the interval \( 0 \leq x \leq L \) with Dirichlet boundary conditions \( f(0) = f(L) = 0 \). \( f(x) \) can be written as an infinite sum of sine terms:
\[
    f(x) = \sum_{n=0}^{\infty} F_n \sqrt{\frac{2}{L}} \sin \left( \frac{\pi nx}{L} \right). \tag{25}
\]

Differentiating this sum gives us
\[
    -\frac{d^2 f(x)}{dx^2} = \sum_{n=1}^{\infty} \left( \frac{n\pi}{L} \right)^2 F_n \sqrt{\frac{2}{L}} \sin \left( \frac{\pi nx}{L} \right). \tag{26}
\]
Defining \( B_n \equiv \left( \frac{n\pi}{L} \right)^2 F_n \), results in

\[
- \frac{d^2 f(x)}{dx^2} = \sum_{n=1}^\infty B_n \sqrt{\frac{2}{L}} \sin \left( \frac{n\pi x}{L} \right).
\]  

(27)

Now our function is in the form

\[
f(x) = \sum_{n=0}^\infty \left( \frac{L}{n\pi} \right)^2 \sqrt{\frac{2}{L}} \sin \left( \frac{n\pi x}{L} \right) B_n,
\]  

(28)

and

\[
B_n = \int_0^L b(y) \sqrt{\frac{2}{L}} \sin \left( \frac{n\pi y}{L} \right) dy.
\]  

(29)

Expanding \( B_n \) in our equation for \( f(x) \) gives us

\[
f(x) = \int_0^L \sum_{n=0}^\infty \frac{2L}{(n\pi)^2} \sin \left( \frac{n\pi x}{L} \right) \sin \left( \frac{n\pi y}{L} \right) b(y) dy.
\]  

(30)

Our goal is to put this equation in the same form as Equation 16:

\[
f(x) = \int_0^L G(x,y) b(y) dy.
\]

This establishes the equality

\[
G(x,y) = \sum_{n=0}^\infty \frac{2L}{(n\pi)^2} \sin \left( \frac{n\pi x}{L} \right) \sin \left( \frac{n\pi y}{L} \right).
\]  

(31)

We may use a contour integrals

\[
\int_{C_1} \frac{F(z)}{\sin z} dz = 2\pi i \sum_{n=1}^\infty (-1)^n F(n\pi) \text{ and } \int_{C_2} \frac{F(z)}{\sin z} dz = 2\pi i \sum_{n=1}^\infty (-1)^n F(n\pi).
\]  

(32)

Contours \( C_1 \) and \( C_2 \) are paths around the poles at positive and negative values on the real axis. We can now write Equation 31 as

\[
\sum_{n=0}^\infty \frac{2L}{(n\pi)^2} \sin \left( \frac{n\pi x}{L} \right) \sin \left( \frac{n\pi y}{L} \right) = \frac{1}{4\pi^2 i} \int_{C_1+C_2} \frac{2L}{z^2 \sin z} \sin \left( \frac{z x}{L} \right) \sin \left( \frac{z y}{L} \right) e^{\pm iz} dz.
\]  

(33)

Before we can apply the theorem of residues, we must be able to close the contour. This closed contour as shown in figure below:
\[ I = \frac{1}{4\pi i} \int_{C_1+C_2} \frac{2L}{z^2 \sin z} \sin \left( \frac{zx}{L} \right) \sin \left( \frac{zy}{L} \right) e^{\pm iz} \, dz. \] (34)

We can see that $\frac{1}{\sin z} \to e^{-R}$ and $\frac{1}{z^2} \to \frac{1}{R^2}$ on the outer contour. In order for our integral on the outside radius to be closed, the rest of the terms must blow up less than these terms will reduce. Expanding the sine terms and distributing according to $e^{\pm iz}$ terms in our integral leaves us with

\[ I = \frac{L}{8\pi i} \oint z^2 \sin z \left( e^{iz(x+y)/L} + e^{-iz(x+y)/L} - e^{iz(x-y)/L} e^{-iz(x-y)/L} \right) \] (35)

This integral now reduces to

\[ I = \frac{1}{2\pi i} \oint \frac{(L-x)y}{\sin z} \, dz \] (36)

We are now able to apply the theorem of residues ($y \leq x$), leaving us with an integral which evaluates to

\[ I = \frac{(L-x)y}{L}. \] (37)

This is exactly the function we had found before.