Linear ODE’s of the Second Order

Linear ODE’s of the Second Order

Prerequisites

General information

The second order linear ODE has the most general form given by the following.

\[ A(x)y'' + B(x)y' + C(x)y = D(x) \]  

(1)

We will work with these equation in a ”standard form” where we set the leading order derivative to have no function of the independent variable attached.

\[ y'' + P(x)y' + Q(x)y = R(x) \]  

(2)

We have rewritten the equation by dividing through by \( A(x) \).

When the function \( R(x) \) is 0 the ODE is said to be **homogeneous**, when it is not the ODE is **inhomogeneous**.

The most important and useful characteristic of these equations is that they are linear. That is to say given any two solutions one can form an infinite family of solutions via linear combination.

Well Posed Problems

An ODE by itself is not sufficient to understand the system it corresponds to. The differential equation only describes the evolution of the system, In order to determine the solution we require sufficient initial conditions or boundary conditions. For the following discussion we will consider ODE’s with initial conditions given by.

\[ y(0) = C \]  

(3)

\[ y'(0) = K \]  

(4)

One can understand the requirement of initial conditions as accounting for the constants of integration that are necessary for solving the ODE.

Boundary conditions allow us to place restrictions on the state space of the solutions. For example if were were modeling a instrument string we could demand.

\[ y(0) = y(L) = 0 \]

These conditions ”pins down” the solution at these points to mirror the system we are trying to model.

Other useful boundary conditions include periodic boundary conditions like the following.

\[ y(x) = y(x + 2\pi) \]

These allow us to model systems on connected domains like around a circle.
Uniqueness

Differential equations alone are limited when it comes to describing a system. For example take the ODE

\[ y'' + y = 0 \]

This system can be solved in any number of ways.

\[ y(x) = \sin(x) \]
\[ y(x) = \cos(x) \]
\[ y(x) = e^{ix} \]

Which of these is the best solution to the problem? With a selection of initial conditions some solutions can be cleaner than others but can we always rely on a certain equation to solve the problem? Luckily ODE’s come pre-packaged with a uniqueness theorem that resolves this issue.

**Theorem:** Consider the initial value problem

\[ y'' + p(t)y' + q(t)y = r(t), \quad y(t_0) = y_0, \quad y'(t_0) = y'_0 \]

If the functions \( p, q, \) and \( r \) are continuous on the interval \( I : \alpha < t < \beta \) containing the point \( t = t_0 \). Then there exists a unique solution \( y = \phi(t) \) of the problem, and that this solution exists throughout the interval \( I \).

This theorem does not routinely crosses ones mind when working with ODE’s in a physical context but, I believe it is an important motivator for the Wronskien. The uniqueness theorem is one of the main reasons that Wronksian analysis works and we will see why momentarily.

Wronskian analysis

**Motivation**

The goal of Wronskian analysis is to streamline the process for finding complete unique solutions to ODE problems. As shown above, there can be numerous ways to solve an ODE but the numerous answers are not always ready to reveal themselves easily. The utility of the Wronskian is that It allows us to find additional solutions to both homogeneous and inhomogeneous versions of a system. We know from the uniqueness theorem that if our existing solution cannot accommodate certain initial conditions then we are missing a piece. The Wronskian uses the linearity of the system to find those other solutions.

**The Wronskian**

The Wronskian of an ODE is given by

\[
W (f_1, \ldots, f_n) (x) = \begin{vmatrix} f_1(x) & f_2(x) & \cdots & f_n(x) \\ f'_1(x) & f'_2(x) & \cdots & f'_n(x) \\ \vdots & \vdots & \ddots & \vdots \\ f_1^{(n-1)}(x) & f_2^{(n-1)}(x) & \cdots & f_n^{(n-1)}(x) \end{vmatrix}
\]

Where the \( f_n \) are solutions to the ODE. We require \( n \) equations for a \( n \)th order ODE.

One can see that if any two solutions are linearly dependent than the Wronskian vanishes.
Consider the second order differential equation.

\[ y'' + py' + qy = 0 \]

where \( p(x), q(x) \) are known functions of \( x \) and \( y(x) \) is the yet to be determined function. Let us call \( y_1, y_2 \) the two solutions of the equation and form their Wronskian.

\[ W(x) = y_1 y_2' - y_2 y_1' \]

Then differentiating \( W(x) \) and using the fact that \( y_i \) obey the above differential equation shows that

\[ W''(x) = P(x)W(x) \]

Therefore, the Wronskian obeys a simple first order differential equation and can be exactly solved:

\[ W(x) = w_0 e^{-\int p(x)dx} \]

Now suppose that we know one of the solutions, say \( y_2 \). Then, by the definition of the Wronskian, \( y_1 \) obeys a first order differential equation:

\[ y'_1 - \frac{y_2'}{y_2} y_1 = \frac{-W(x)}{y_2} \]

and can be solved exactly.

The method is easily generalized to higher order equations.

**The Method of Variation of Constants**

Despite its paradoxical name this method can be vary useful. Using the same notation as before.

\[ y'' + P(x)y' + Q(x)y = R(x) \]

Suppose we have a solution to the inhomogeneous case.

\[ \hat{y}(x) \]

Then we have generally

\[ y(x) = \hat{y}(x) + c_1 y_1(x) + c_2 y_2(x) \]

This satisfies the full solution because the two \( y \)’s evaluate to zero. Thus given sufficient initial conditions we can solve any initial value problem. To find solutions to the inhomogeneous solution. Consider the solution of form.

\[ \hat{y}(x) = c_1(x) y_1(x) + c_2(x) y_2(x) \]

Here we have made the constants a function of \( x \). This is the method of variation of constants. The functions \( c \) will be arbitrary. That is to say that they have no structure related to the solutions so we are free to implement our own restrictions on them.

We make the following constraint.

\[ c_1'(x)(x) + c_2'(x)y_2(x) = 0 \]
The we can plug these new solutions into the equation to recover a new constraint.

\[ c_1'(x)y_1(x) + c_2'(x)y_2(x) = R(x) \]

So we have two constraints.

\[ c_1'(x)y_1(x) + c_2'(x)y_2(x) = 0 \]
\[ c_1'(x)y_1'(x) + c_2'(x)y_2'(x) = R(x) \]

These will help us find our solution. We find

\[ c_1'(x) = -\frac{R(x)y_2}{|W|} \]
\[ c_2'(x) = \frac{R(x)y_1}{|W|} \]

Recalling

\[ W(x) = w_0e^{-\int_{x_0}^{x} P(x)dx} = |W| \]

We can find the functions c via integration.

\[ c_1(x) = -\int_{x_0}^{x} \frac{R(x)y_2(x)}{W(x)} \]
\[ c_2(x) = \int_{x_0}^{x} \frac{R(x)y_1(x)}{W(x)} \]

Knowing this completes our knowledge of the equation that solves the system. The arbitrariness in \( w_0 \) and \( x_0 \) is accounted for in the initial conditions

**Example**

Consider the simple forced harmonic oscillator

\[ y''(x) + y(x) = f(x) \]

Let

\[ y_1 = \cos(x) \]
\[ y_2 = \sin(x) \]

Then

\[ c_1(x) = -\int_{x_0}^{x} f(x) \sin(x) \]
\[ c_2(x) = \int_{x_0}^{x} f(x) \cos(x) \]

Thus we have the general solution.

\[ \hat{y}(x) = c_1(x)y_1(x) + c_2(x)y_2(x) \]