09/16/19 - Lecture 3:

- **Review of Complex Numbers:**
  - **General Definitions:**
    * $r \equiv \text{Modulus} = |z|$ (Magnitude = Absolute Value)
    * $\varphi \equiv \text{Argument} = \arg(z)$ (Phase)
    * Cartesian Expression: $z = x + iy$
    * Polar Expression: $z = re^{i\varphi}$

- **Conjugation and Inversion of a Complex Number:**
  * Conjugate $z^* = x - iy = re^{-i\varphi}$
  * Modulus $zz^* = |z|^2 = x^2 + y^2 = r^2$
  * Inverse $z^{-1} = \frac{z^*}{|z|^2} = \frac{x - iy}{r^2}$

- **Trigonometric Identities:**
  * Euler’s Formula $e^{i\varphi} = \cos\varphi + i\sin\varphi$
  * Euler’s Identity $e^{i\pi} = -1$
  * $\sin(\alpha + \beta) = \sin\alpha\cos\beta + \sin\beta\cos\alpha$
  * $\cos(\alpha + \beta) = \cos\alpha\cos\beta - \sin\alpha\sin\beta$
  * De Moivre’s Formula $(\cos\varphi + i\sin\varphi)^n = \cos(n\varphi) + i\sin(n\varphi)$

- **Equivalent Expressions of Some Complex Functions:**
  - In General:
    * $f(x, y) \ Leftrightarrow f(z) \ \forall \ z$
    * $f(x, y) \ Leftrightarrow f(z, z^*) \ \forall \ z$
    * For some functions, like $f(x, y) \in \{x^2, xy, y^2\}$, we cannot construct a function with only $z$ to represent them.
* Ex: \( x^2 = \frac{1}{4}(z + z^*)^2 \) for:
  
  \[
  \cdot z^2 = x^2 - y^2 + 2ixy \\
  \cdot zz^* = x^2 + y^2 \\
  \cdot (z^*)^2 = x^2 - y^2 - 2ixy
  \]

- Series Expansion of \( n \)th Degree Polynomial about \( c \):
  \[ P_n(z) |_{c} = \sum_{k=0}^{n} \left( \frac{(z-c)^k}{k!} \right) \]

- Geometric Series:
  \[ f(z) = \frac{1}{1-z} = 1 + z + z^2 + z^3 + \ldots \]

### Analytic Functions:

- **Derivation:** Let \( f(x, y) = u(x, y) + v(x, y) \)

  * Total Derivative of \( f \):
    \[
    df = \frac{\partial u}{\partial x}dx + \frac{\partial u}{\partial y}dy + i\frac{\partial v}{\partial x}dx + i\frac{\partial v}{\partial y}dy
    \]

  * Rearranging terms:
    \[ \Rightarrow (\frac{\partial u}{\partial x} + i\frac{\partial v}{\partial x})dx + (\frac{\partial u}{\partial y} + i\frac{\partial v}{\partial y})dy = df \]

  * Constraint: If \( f(x, y) \) is a Holomorphic & Analytic function, then we expect it to be complex differentiable around any point within its domain. Thus this condition for analyticity yields:
    \[ df(z) = f'(z)dz = f'(z)(dx + idy) = f'(z)dx + f'(z)idy \]

  * For \( f(z) = f(x, y) \) we can match the total derivative to the constraint above, such that it is apparent:
    \[ (\frac{\partial u}{\partial x} + i\frac{\partial v}{\partial x}) = (\frac{\partial u}{\partial y} + i\frac{\partial v}{\partial y}) \]

  * By orthogonality of the real & imaginary parts of a complex function, we now return a set of equations that are an equivalent representation of the original condition of analyticity:
    \[ \frac{\partial u}{\partial x} = \frac{\partial v}{\partial y} \quad \& \quad \frac{\partial u}{\partial y} = -\frac{\partial v}{\partial x} \quad \Rightarrow \quad \text{These are known as the Cauchy - Riemann Conditions} \]

- By satisfying the Cauchy-Riemann conditions, we imply:
  
  * \( u \) & \( v \) are harmonic conjugates of one another.
  * \( f(z) \) is complex differentiable/holomorphic
  * \( f(z) \) is analytic

### Integration of a Complex Function:

For a domain \( \Sigma \) bounded by the path \( \gamma \) in the complex plane, we can define:

\[
I = \oint_{\gamma} f(z)dz = \int_{S_0}^{S_1} [u(s) + iv(s)]\left( \frac{\partial x}{\partial s} ds + i\frac{\partial y}{\partial s} ds \right)
\]
- For $I = \text{Re}[I] + i\text{Im}[I]$

\[
\begin{align*}
\text{Re} &= \int_{\gamma} (u \frac{\partial x}{\partial s} - v \frac{\partial y}{\partial s}) ds = \int \vec{w}_1 \cdot d\vec{r} \\
\text{Im} &= \int_{\gamma} (v \frac{\partial x}{\partial s} + u \frac{\partial y}{\partial s}) ds = \int \vec{w}_2 \cdot d\vec{r}
\end{align*}
\]

- We apply Stoke’s Thm, and transform to integral over area $\Sigma$ bounded by $\gamma$. We then plugin for the value of the curl:

\[
\begin{align*}
\text{Re} &= \int_{\Sigma} \epsilon \text{curl} (\vec{w}_1) d\sigma = \int \left(\frac{\partial v}{\partial x} - \frac{\partial u}{\partial y}\right) d\sigma = 0 \\
\text{Im} &= \int_{\Sigma} \epsilon \text{curl} (\vec{w}_2) d\sigma = \int \left(\frac{\partial u}{\partial x} - \frac{\partial v}{\partial y}\right) d\sigma = 0
\end{align*}
\]

- Thus we arrive at the conclusion that $I = \text{Re}[I] + i\text{Im}[I] = 0$, such that;

\[\oint_{\gamma} f(z) dz = 0\]

(This is a key form of Cauchy’s Integral Theorem. The Thm applies wherever the function $f(z)$ is analytic inside the boundary $\gamma$)

\[\rho \quad \gamma \quad \Sigma \quad C \quad z_0 \quad x \quad y \quad Z\]

- What if there is a point or a discrete set of points within $\gamma$ where the complex function is not analytic?

Ex: For analytic function $f(w)$, let $g(w) = \frac{f(w)}{w-z_0}$.

We see at $w = z_0$ that the function is no longer analytic. If the path $\gamma$ is simply connected, then to handle this case we can make a new path by shrink the path $\gamma$ into a path $C$ that is near $z_0$. However we can join the path $\gamma$ and $C$ such that only the region where $g(w)$ is analytic is enclosed, and $z_0$ is not enclosed.

- Thus, $I = \oint_{\gamma} g(w) dw + \oint_{C} g(w) dw = 0$

- If we let $w = z + \epsilon e^{-i\phi}$ for small $\epsilon$, we can compute the residue from closed loop $C$ of the pole at $z_0$, s.t. $\oint_{C} \frac{f(w)}{w-z_0} dw = 2\pi i f(z_0)$ $\implies$ This is the full form of Cauchy’s integral theorem, for analytic function $f(w)$.
• Series Expansion of $f(z)$:

$$f(z) = \frac{1}{2\pi i} \oint_{\gamma} \frac{f(w)}{w-a} \frac{1}{w-a} dw = \frac{1}{2\pi i} \oint_{\gamma} \frac{f(w)}{(w-a)^2} \frac{1}{w-a} dw$$

- So for $z$ inside the circle, $w > z \implies \left( \frac{z-a}{w-a} < 1 \right)$ s.t. we can apply the geometric series to evaluate.

- We can rewrite this as:

$$f(z) = \sum_{n=0}^{\infty} c_n (z - a)^n$$

$$c_n = \frac{1}{2\pi i} \oint_{\gamma} \frac{f(w)}{(w-a)^{n+1}} dw$$

**09/18/19 - Lecture 4:**

Motivation: Give some function $f(z)$ which is analytic over some area save for a singularity at point $A$ within that area, how would one expand this function?

1. Take advantage of Cauchy’s Integral Theorem (seen above).

2. $\oint_{\gamma} \frac{f(w)}{w-z} dw = 2\pi i f(z)$

3. Cleverly substitute the $w - z$ term for $w - a - x - a$ then manipulate to get $(w - a)(1 - \frac{z-a}{w-a})$

   (a) Notice that $w - a > z - a$ so one can expand the $\frac{z-a}{w-a}$ term using a Taylor series.

4. $f(z) = \frac{1}{2\pi i} \oint_{\gamma} \frac{f(w)}{w-a} (1 + \frac{z-a}{w-a} + (\frac{z-a}{w-a})^2 + ...) dw$

5. $f(z) = \sum_{n=0}^{\infty} \frac{(z - a)^n}{n+1} \oint_{\gamma} \frac{f(w)}{(w-a)^{n+1}} dw$

6. $f(z) = \sum_{n=0}^{\infty} C_n (z - a)^n$ where $C_n = \frac{1}{2\pi i} \oint_{\gamma} \frac{f(w)}{(w-a)^{n+1}} dw$
7. \[
\begin{align*}
\oint_{\gamma} f(z)dz & \leftrightarrow -\oint_{\gamma'} f(z)dz \\
\oint_{\gamma} f(z)dz & \leftrightarrow \oint_{\gamma'} f(z)dz
\end{align*}
\]
(a) The paths $\gamma$ and $\gamma'$ above do not need to be regular they just need to be simply connected.

8. Applying this to some analytic function with a singularity, for instance $f(z) = \frac{1}{z-z^2} = \frac{1}{z(1+z)} = \frac{1}{2}(1 + z + z^2 + ...)$ one can exclude the singularity.

9. $f(z) = \frac{1}{2\pi i} \oint_{C} f(w) \frac{1}{w-z} + \frac{1}{2\pi i} \oint_{c} f(w) \frac{1}{w-z}$ ($\gamma$ and $\gamma'$ have been exchanged for $C$ and $c$ respectively.

10. $\frac{1}{2\pi i} \oint_{C} f(w) \frac{1}{w-z} = \sum_{n=0}^{\infty} C_n (z-a)^n$

11. $\frac{1}{2\pi i} \oint_{c} f(w) \frac{1}{w-z}$

\[
= -\frac{1}{2\pi i} \oint_{c} f(w) \frac{1}{z-w} \\
= \frac{1}{2\pi i} \oint_{c'} f(w) \frac{1}{z-w} \\
= \frac{1}{2\pi i} \oint_{c'} f(w) \frac{1}{(z-a)-(w-a)} \\
= \frac{1}{2\pi i} \oint_{c'} f(w) \frac{1}{(z-a)(1-\frac{w-a}{z-a})} (\text{Now we can do the expansion.})
\]

\[
= \sum_{n=-1}^{\infty} \left(\frac{1}{z-a}\right)^n \frac{1}{2\pi i} \oint_{c'} f(w) \frac{1}{(w-a)^{n+1}} dw
\]

12. This means the function can be expanded as $f(z) = \sum_{n=0}^{\infty} C_n (z-a)^n$ with $C_n = \frac{1}{2\pi i} \oint_{\gamma} \frac{f(w)}{(w-a)^{n+1}} dw$

(a) Note: We have switched the path notation back to $\gamma$. In this final equation we have taken the paths $C$ and $c$ to be overlapping, thus merging the inner and outer circles into one common path $\gamma$.

13. This is known as the Laurent Series Expansion.
14. The Laurent Series Expansion can be expressed in two paths:
\[
\begin{align*}
&\sum_{n=0}^{\infty} C_n(z-a)^n \quad \text{The regular path} \\
&\sum_{n=-\infty}^{-1} C_n(z-a)^n \quad \text{The singular path}
\end{align*}
\]

(a) If the singular path vanishes, then the Laurent series goes to a Taylor series.

15. If the singular path vanishes, all the coefficients \( C_n \) in the singular path go to zero.

16. \( C_0 = C_1 = \ldots = C_{m-1} = 0 \)

17. \( f(z) = (z-a)^m \sum C_m + C_{m+1}(z-a) \) (Zero of order \( m \))

(a) \( \sum C_m + C_{m+1}(z-a) = g(z) \)

18. \( \sum_{n=1}^{\infty} C_n(z-a)^n = \frac{1}{(z-a)^m} g(z) \) (Pole of order \( m \))

19. For the function \( f(z) = \frac{1}{z^2} \) there are two simple poles at \( z = 0 \) and \( z = 1 \) (from \( f(z) = \frac{1}{z(1-z)} \)).

20. If all \( C_n \) are non-zero for the singular path, the function \( f(z) \) has an essential singularity at \( z = a \).

21. The function \( e^{\frac{1}{z}} \) has an essential singularity at \( z = 0 \) because the expansion, \( e^{\frac{1}{z}} = 1 + \frac{1}{z} + \frac{1}{2z^2} + \frac{1}{3!z^3} + \ldots \) contains singularities for every power of \( z \).

22. Taking the limit as \( z \to 0^+ \) the function approaches infinity and as \( z \to 0^- \) the function approaches zero.

23. Now we consider how the using the Residue will simplify our handling of singularities in analytic functions. The Residue is the coefficient of the pole term.

24. \( \frac{1}{2\pi i} \oint \gamma f(z) dz = Res(f,a) \)

In words: the path integral of an analytic function around \( \gamma \) is equal to the residue of the function at \( a \), the singularity.

25. For a function with more than 1 singularity:
\( \oint \gamma f(z) dz = 2\pi i \sum_{i=1}^{n} Res(f,a_i) \)

where \( a_i \) are the individual singularities.

26. This is known as the Theorem of Residues.