Inverting these equations we can express v_h as a function of p_h, q_h and t. Substituting back in Eq. 29 we find

$$33)\qquad S\ (q_h,\ t) = S\ (q_h, t) - \Sigma_h\ v_h\ (p,\ q,\ t)\ p_h\ \Delta t -$$

$$- \frac{\partial S}{\partial t}\,\Delta t + L\ (v_h\ (p,\ q,\ t),\ q_h,\ t)\ \Delta t\ .$$

or

$$34)\qquad - \frac{\partial S}{\partial t} = \Sigma_h\ v_h\ (p,\ q,\ t)\ p_h - L\ (v_h\ (p,\ q,\ t),\ q_h,\ t)\ .$$

Notice that the expression

$$35)\qquad H(p_h,\ q_h,\ t) = \Sigma_h\ v_h\ p_h - L\ (v_h,\ q_h,\ t)$$

is, by definition, the hamiltonian of the system. Eq. 34 can then be written

$$36)\qquad - \frac{\partial S}{\partial t} = H\left(\frac{\partial S}{\partial q_h},\ q_h,\ t\right)$$

and this is the equation of Hamilton-Jacobi: it says that the time derivative of the minimal action S(q,t) equals the negative of the hamiltonian, where the momenta have been replaced by $\frac{\partial S}{\partial q_h}$.

The equation of Hamilton-Jacobi is a very important equation, and it is impossible to give a good account of its meaning and application in these short notes. Let us consider the form it takes for the simple system we used to introduce it, with lagrangian

37) $$L = \frac{\beta}{2} v^2 - gq .$$

The equation of Hamilton-Jacobi is given by Eq. 26 (we use again units such that $\beta = 1$):

38) $$-\frac{\partial S}{\partial t} = \frac{1}{2}\left(\frac{\partial S}{\partial q}\right)^2 + gq .$$

It is easy to check that this equation is solved by

39) $$S = -\frac{g^2 t^3}{24} - \frac{gtq}{2} + \frac{q^2}{2t} .$$

Differentiating we find

40) $$\frac{\partial S}{\partial q} = -\frac{gt}{2} + \frac{q}{t} ,$$

and Eq. 23 gives us

41) $$\frac{dq}{dt} = -\frac{gt}{2} + \frac{q}{t} .$$

This equation can also be integrated, and gives

$$q = -\frac{gt^2}{2} + v_o t , \tag{42}$$

where v_o is an arbitrary constant. We see that the equation of Hamilton Jacobi, together with the relation $p = \beta v = \frac{\partial S}{\partial q}$, produces a family of trajectories, passing through the origin at $t = 0$.

This may seem a complicated way to solve a simple problem, but, as we shall soon see, some of the operations used to find the trajectories can be replaced by algebraic operations, and then the method becomes much more powerful.

(Incidentally, it is interesting to compare the results given by Eq's 39 and 42 with what we found in the discrete case. Adjusting v_o so that $q = 20$ at $t = 4$ we find

$$q = -\frac{gt^2}{2} + 2gt + 5t .$$

This gives $q(1) = 8$, $q(2) = 14$, $q(3) = 18$ and $S = -40.67$ for $g = 2$; $q(1) = 12.5$, $q(2) = 20$, $q(3) = 22.5$ and $S = -216.67$ for $g = 5$. The coincidence of the $q(i)$ with the q_i, averaged over the degenerate policies for $g = 5$, is fortuitous, and depends on the choice of integer parameters, but the numbers do show the close correspondence between the two problems.)

In general, any solution $S(q_h,t)$ of the equation of Hamilton-Jacobi determines a family of trajectories, which can be found integrating the equations

$$\frac{\partial L}{\partial v_h}\left(v_h = \frac{dq_h}{dt}, q_h, t\right) = \frac{\partial}{\partial q_h} S(q_h, t) . \tag{43}$$

If S represents the minimal action to go from an initial configuration q_h° at $t = t_o$ to q_h at t, all the trajectories, integrated backwards in time, will converge to q_h° for $t = t_o$. However, there will be solutions which determine families of trajectories, that do not converge to any definite point at any instant of time.

Some of these solutions are extremely relevant, even if it is not possible to attach to them the meaning of the minimal action to go from a given point q_h° at t_o to a generic point q_h at t. To see that the convergence property is not required for a physical interpretation, just suppose that all trajectories did pass through a common point at a very remote time -T and that some arbitrary time dependent Lagrangian $\tilde{L}(v_h, q_h, t)$ governed the evolution of the system for $-T \leqslant t \leqslant t_o$. At $t = t_o$ our Lagrangian $L(v_h, q_h, t)$ takes over, and determines the further evolution of the trajectories. If we use now $L(v_h, q_h, t)$ to evaluate S also for $t < t_o$, and to extrapolate the trajectories backwards in time accordingly, in general the trajectories will never converge to a single point, and still they represent possible motions of the system.

With this in mind, let us show how the solutions to the equation of Hamilton Jacobi may be used to find the trajectories through algebraic equations.

The crucial point is that, if we succeed in finding a class of solutions depending on one arbitrary parameter α,

44) $$S = S(q_h, t, \alpha) ,$$

the condition

45) $$\frac{\partial S}{\partial \alpha}(q_h, t, \alpha) = \text{const} = \beta$$

is compatible with the equations of motion.

Let us prove this statement. The equations of motion for the trajectories are given by Eq. 43. Let us check the compatability of Eq. 45 with Eq. 43 by differentiating Eq. 45 with respect to time. We find the condition

46) $$\frac{d}{dt}\frac{\partial S}{\partial \alpha}(q_h, t, \alpha) =$$

$$= \Sigma_h \frac{\partial^2 S}{\partial \alpha \partial q_h}\dot{q}_h + \frac{\partial S}{\partial \alpha \partial t} = 0 ,$$

or

47) $$\frac{\partial}{\partial \alpha}\left\{ \Sigma_h \frac{\partial S}{\partial q_h} v_h + \frac{\partial S}{\partial t} \right\} = 0,$$

where the derivative $\frac{\partial}{\partial \alpha}$ acts only on the explicit dependence of S on α. We substitute $\frac{\partial S}{\partial t} = - H\left(\frac{\partial S}{\partial q_h}, t\right)$,

and find

48) $$\Sigma_h \left\{ \frac{\partial S}{\partial \alpha \partial q_h} v_h - \frac{\partial H}{\partial \frac{\partial S}{\partial q_h}} \frac{\partial S}{\partial \alpha \partial q_h} \right\} = 0.$$

But the definition of H (see Eq. 35) implies that

$$49) \qquad v_h = \frac{\partial H}{\partial \frac{\partial S}{\partial q_h}} ,$$

so that Eq. 48 is indeed satisfied.

Eq. 45 is a very powerful tool: if we can find a class of solutions with a number of independent parameters α_h equal to the number of degrees of freedom of the system (we do not count the trivial additive constant that can always be added to S as one of the independent parameters), we can impose the conditions

$$50) \qquad \frac{\partial S}{\partial \alpha_h}(q_h, t, \alpha_h) = \beta_h = \text{const}$$

without violating the equations of motion. The set of Eq's 50 in general admits a single solution

$$51) \qquad q_h = q_h (\alpha_h, \beta_h, t) ,$$

and we see that we can find the trajectories by solving a system of algebraic equations.

Let us apply these techniques to the solution of the Kepler problem in the plane. We take as lagrangian in polar coordinates

$$52) \qquad L = \frac{m}{2} (\dot{r}^2 + r^2\dot{\phi}^2) + \frac{k}{r} .$$