PY 408 - Intermediate Mechanics

Midterm exam # 2 - November 10, 2011.

Solve all three problems. Give a clear explanation of your work. Correct solutions will be credited with 30 points for problem 1, and 35 points for problems 2 and 3. Points will be taken away for errors, inaccuracies and poor or missing explanation of the work.

Problem 1

Write the mathematical expression for the Poisson bracket $[A, B]$ of two functions $A(p_j, q_j), B(p_j, q_j)$ of coordinates and momenta.

Then for each of the changes of variables specified below state whether it is a canonical transformation. Justify your answers.

1:

$$P = \alpha p \qquad Q = \alpha q \tag{1}$$

2:

$$P = -q \qquad Q = p \tag{2}$$

3:

$$P = \alpha p \qquad Q = \frac{q}{\alpha} \tag{3}$$

4:

$$P = q \qquad Q = p \tag{4}$$

5:

$$P = \frac{p^2}{2m} \qquad Q = \frac{mq}{p} \tag{5}$$

($\alpha \neq 1$ and m are non-zero constants)

Solution

$$[A, B] = \sum_j \Big(\frac{\partial A}{\partial p_j}\frac{\partial B}{\partial q_j} - \frac{\partial A}{\partial q_j}\frac{\partial B}{\partial p_j}\Big) \tag{6}$$

1:

$$[P, Q] == \alpha^2[p, q] = \alpha^2 \neq 1 \tag{7}$$

The change of variables is not a canonical transformation. 2:

$$[P, Q] = -[q, p] = [p.q] = 1 \tag{8}$$

The change of variables is a canonical transformation. 3:

$$[P, Q] = \frac{\alpha}{\alpha}[p, q] = 1 \tag{9}$$

The change of variables is a canonical transformation. 4:

$$[P, Q] = [q, p] = -[p.q] = -1 \tag{10}$$

The change of variables is not a canonical transformation. 5:

$$P = \frac{p^2}{2m} \qquad Q = \frac{mq}{p}[P, Q] = \frac{\partial(p^2/2m)}{\partial p}\frac{\partial(mq/p)}{\partial q} = \frac{p}{m}\frac{m}{p} = 1 \tag{11}$$

The change of variables is a canonical transformation.

Problem 2

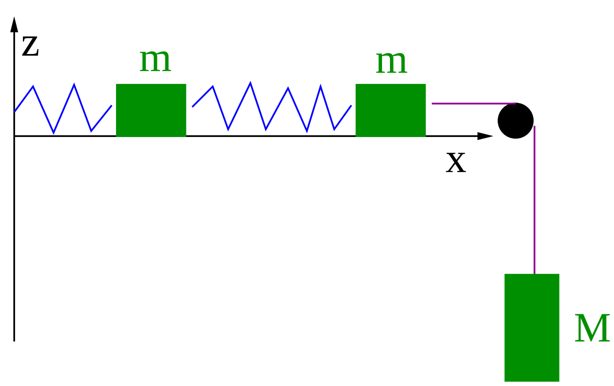

Figure 1: Illustration for problem 2.

Consider the mechanical system illustrated in Fig. 1: two objects of mass m move without friction along the x-axis. The object to the left is connected to two springs: one spring has the other end fixed at $x = 0$, the other spring is also connected to the object to the right. The object to the right is connected to the spring mentioned above and to an inextensible rope which goes over a pulley and is attached at the other end to a weight of mass M. The two springs have the same spring constant k. The springs, pulley and rope have negligible mass.

Take as generalized coordinates q_1, q_2 the difference between the x-coordinates of the bodies and the values $x_1^{(0)}, x_2^{(0)}$ of their x-coordinates when in equilibrium

$$\begin{aligned} q_1 &= x_1 - x_1^{(0)} \\ q_2 &= x_2 - x_2^{(0)} \end{aligned} \tag{12}$$

Write down the Lagrangian of the system and Lagrange's equations of motion. Find then the angular frequencies of the two normal modes of oscillation of the system, taking $M = 3m$ in order to simplify the algebra in this last part of the problem.

(Note: the rest length of the springs and the length of the rope are not given because they are not needed for the solution of this problem.)

Solution

For definiteness we label the object to the left with index 1, the one to the right with index 2. The magnitude of the velocity of the weight is the same as the magnitude of the velocity of object 2. Thus the kinetic energy is

$$K = \frac{m\dot{q}_1^2 + (m+M)\dot{q}_2^2}{2} \tag{13}$$

The potential energy $V(q_1, q_2)$ will only contain terms quadratic in q_1, q_2 and a constant term, since the linear terms must vanish on account that the system is in equilibrium for $q_1 = q_2 = 0$. Neglecting a possible irrelevant constant V will thus be given by

$$V = \frac{kq_1^2 + k(q_2 - q_1)^2}{2} \tag{14}$$

From the Lagrangian

$$L = \frac{m\dot{q}_1^2 + (m+M)\dot{q}_2^2}{2} - \frac{kq_1^2 + k(q_2 - q_1)^2}{2} \tag{15}$$

we derive the equations of motion

$$\begin{aligned} m\ddot{q}_1 &= -k(2q_1 - q_2) \\ (m+M)\ddot{q}_2 &= -k(q_2 - q_1) \end{aligned} \tag{16}$$

We look for normal modes of oscillation of the type

$$q_i(t) = c_i e^{\imath\omega t} \tag{17}$$

Substituting in Eq. 16 with $M = 3m$ we get

$$\begin{aligned} (m\omega^2 - 2k)c_1 + kc_2 &= 0 \\ kc_1 + [4m\omega^2 - k]c_2 &= 0 \end{aligned} \tag{18}$$

which will have non-zero solutions only if

$$(m\omega^2 - 2k)[4m\omega^2 - k] - k^2 = 4m^2\omega^4 - 9mk\omega^2 + k^2 = 0 \tag{19}$$

which has the two solutions

$$\omega_\pm^2 = \frac{k}{m}\,\frac{9 \pm \sqrt{65}}{8} \tag{20}$$

Correspondingly, the angular frequencies of the normal modes will be $\omega_- \approx 0.34\sqrt{k/m}$ and $\omega_+ \approx 1.46\sqrt{k/m}$.

Problem 3

Note: this is the same as problem 3 in assignment 5.

Calculate the trajectory of a point-like object of mass m which moves in the $x - y$ plane subject only to the force which derives from a central potential

$$V(r) = \frac{kr^2}{2} \tag{21}$$

by using polar coordinates r, ϕ. Use conservation of energy and angular momentum to express r as a function of ϕ. Show that the trajectory is an ellipse with center at the origin.

You must solve the problem using polar coordinates throughout, inclusive of the expressions for energy and angular momentum. You should introduce Cartesian coordinates only at the end of the problem to convert the formula giving r as function of ϕ into the equation of an ellipse.

Solution

Please see the posted solution to the problem 3 in assignment 5.