PY 408 - Intermediate Mechanics

Final exam - December 19, 2011.

Solve all four problems. Give a clear explanation of your work. Correct solutions will be credited with 25 points for each problem. Points will be taken away for errors, inaccuracies and poor or missing explanation of the work.

Problem 1

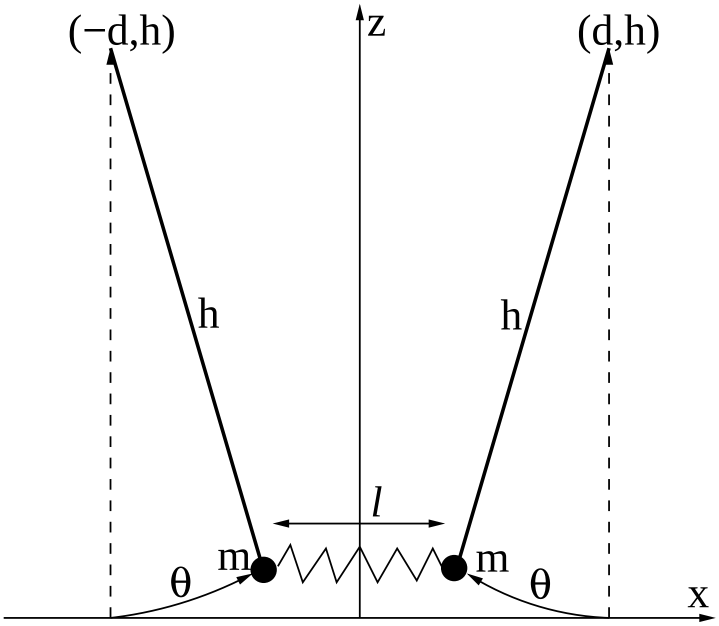

Figure 1: Illustration for problem 1.

Two identical simple pendulums of mass m and length h have their fulcrums at the points P_1 of coordinates (d, h) and P_2 of coordinates $(-d, h)$ (see Fig. 2.) The masses of the two pendulums are connected by a spring of spring constant k and zero rest length, so that if the spring is stretched to length ℓ its potential energy is $k\ell^2/2$. Find the equilibrium values of the angles θ_1, θ_2 which the pendulums form with the vertical either:

i) by the principle of virtual work;

or:

ii) by minimizing the potential energy of the system. (With conservative forces the two are equivalent.)

Important note 1). The system has two degrees of freedom, but given the symmetry you may assume that at equilibrium $\theta_1 = \theta_2 = \theta$ (see Fig. 2 for the orientation of the angles) and thus, if you use the principle of virtual work, you may also assume $d\theta_1 = d\theta_2 = d\theta$.

Important note 2). You should obtain a trigonometric equation which the angle θ satisfies at equilibrium and show that the equation always has one solution in the range $0 \leq \theta \leq \pi/2$. You are not asked to solve the equation, which does not have a simple solution.

Solution

We take $\theta_1 = \theta_2 = \theta$.

The coordinates of the mass in the positive x half-plane are

$$\begin{aligned} x &= d - h\sin\theta && (1) \\ z &= h - h\cos\theta && (2) \end{aligned}$$

i) Using the principle of virtual work.

An infinitesimal change of θ by $d\theta$ produces

$$\begin{aligned} dx &= -h\cos\theta\, d\theta && (3) \\ dz &= h\sin\theta\, d\theta && (4) \end{aligned}$$

Correspondingly the spring does work (considering both pendulums)

$$dW_s = -2k\ell\, dx = -4kx\, dx = 4kh(d - h\sin\theta)cos\theta\, d\theta \qquad (5)$$

while the force of gravity does work

$$dW_g = -2mg\, dz = -2mgh\sin\theta\, d\theta \qquad (6)$$

Demanding $dW_s + dW_g = 0$ gives the equilibrium condition

$$4kh(d - h\sin\theta)cos\theta = 2mgh\sin\theta \qquad (7)$$

or

$$d - h\sin\theta = \frac{mg}{2k}\tan\theta \tag{8}$$

Since the l.h.s. of this equation decreases monotonically from d down to $d-h$ while θ increases from 0 to $\pi/2$, while the r.h.s. increases monotonically from 0 to ∞ while θ varies in the same range, there will always be a single solution with $0 \leq \theta \leq \pi/2$. Also we see from Eq. 8 that if the ratio mg/k grows, which makes the function at the r.h.s. of the equation steeper, then the value of the angle θ which solves the equation decreases, as one intuitively expects.

ii) Minimizing the potential energy.

The potential energy of the string is

$$V_s = \frac{k\ell^2}{2} = 2kx^2 = 2k(d - h\sin\theta)^2 \tag{9}$$

The gravitational potential energy is

$$V_g = 2mgz = 2mg(h - h\cos\theta) \tag{10}$$

Thus the total potential energy is

$$V = V_s + V_g = 2k(d - h\sin\theta)^2 + 2mg(h - h\cos\theta) \tag{11}$$

Demanding that V is at a minimum we get the equation

$$\frac{d}{d\theta}V = -4kh(d - h\sin\theta)\cos\theta + 2mgh\sin\theta = 0 \tag{12}$$

or

$$2k(d - h\sin\theta)\cos\theta = mg\sin\theta \tag{13}$$

which is equivalent to Eq. 8.

Problem 2

A point-like object of mass m moves in the $x-y$ plane and is connected to three massless springs anchored at the vertexes $(d, 0)$, $(-d/2, \sqrt{3}d/2)$, $(-d/2, -\sqrt{3}d/2)$ of an equilateral triangle (see Fig. 2.) The spring anchored at $d, 0$ has spring constant k_1 and zero rest length, so that if its length is ℓ_1, its potential energy is $k_1\ell_1^2/2$. The other two have spring constant k_2 and

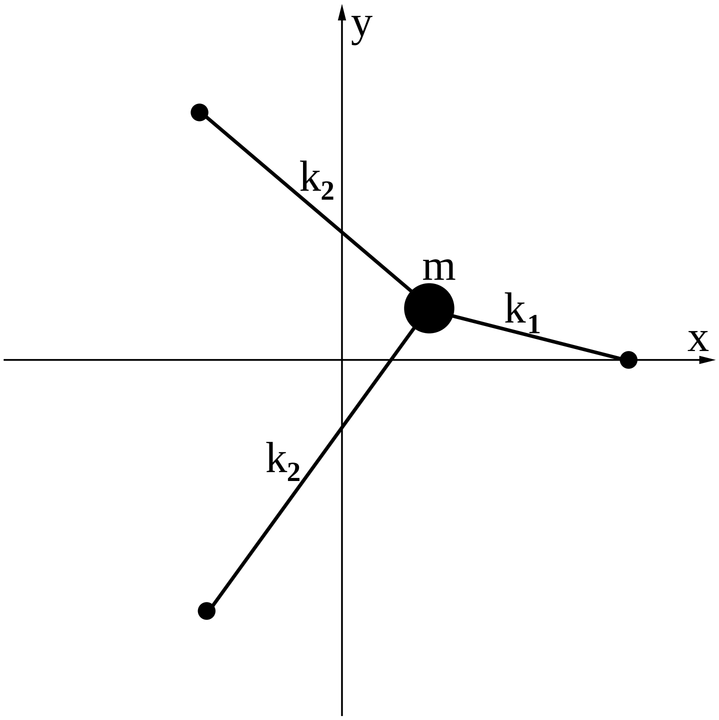

Figure 2: Illustration for problem 2.

also zero rest length. Find the Lagrangian of the system, then derive and solve Lagrange's equations of motion.

Solution

We take the coordinates x, y of the object as the canonical coordinates. The kinetic energy is

$$K = \frac{m(\dot{x}^2 + \dot{y}^2)}{2} \tag{14}$$

The extensions of the three springs are

$$\ell_1 = \sqrt{(d-x)^2 + y^2} \tag{15}$$

$$\ell_2 = \sqrt{(-d/2-x)^2 + (\sqrt{3}d/2-y)^2} \tag{16}$$

$$\ell_3 = \sqrt{(-d/2-x)^2 + (-\sqrt{3}d/2-y)^2} \tag{17}$$

The potential energy is given by

$$V = \frac{k_1[(d-x)^2+y^2]}{2}$$
$$+\frac{k_2[(-d/2-x)^2+(\sqrt{3}d/2-y)^2+(-d/2-x)^2+(-\sqrt{3}d/2-y)^2]}{2} =$$
$$\frac{(k_1+2k_2)(x^2+y^2)-2(k_1-k_2)x+c}{2} \tag{18}$$

where the constant $c = (k_1+2k_2)d^2$ only contributes a constant term to the potential energy and will therefore be neglected.

By minimizing V we find the equilibrium position x_0, y_0 of the mass:

$$x_0 = \frac{k_1-k_2}{k_1+2k_2} \tag{19}$$
$$y_0 = 0 \tag{20}$$

Taking as new canonical coordinates the differences between the x, y coordinates and their equilibrium values

$$q_1 = x - x_0 \tag{21}$$
$$q_2 = y \tag{22}$$

and again neglecting an irrelevant constant term, we find for the Lagrangian

$$L = \frac{m(\dot{q}_1^2+\dot{q}_2^2)}{2} - \frac{(k_1+2k_2)(q_1^2+q_2^2)}{2} \tag{23}$$

The equations of motion are

$$m\ddot{q}_1 = -(k_1+2k_2)q_1 \tag{24}$$
$$m\ddot{q}_2 = -(k_1+2k_2)q_2 \tag{25}$$

But we recognize in Eqs. 23 and 24, 25 the Lagrangian and equations of motion of a harmonic oscillator with mass m and spring constant $k = k_1+2k_2$ moving in the $x-y$ plane. Thus the object in the problem will move of harmonic motion in the plane around the point of equilibrium with angular velocity $\omega = \sqrt{(k_1+2k_2)/m}$. Precisely, the solution will be

$$x(t) = x_0 + A_1\sin(\omega t+\phi_1) \tag{26}$$
$$y(t) = A_2\sin(\omega t+\phi_2) \tag{27}$$

where the amplitudes A_1, A_2 and phases ϕ_1, ϕ_2 will be determined by the initial conditions.

Problem 3

(Please note: this is the same as Problem 1 in assignment 8.)

Consider the motion of a point-like object in one dimension under a constant force, with Hamiltonian

$$H = \frac{p^2}{2m} + mgq \tag{28}$$

A) Verify that the change of variables

$$p = -mgQ \tag{29}$$

$$q = \frac{P}{mg} - \frac{gQ^2}{2} \tag{30}$$

is a canonical transformation.

B) Express the Hamiltonian in terms of the new variables P, Q. Derive and solve the equations of motion for P and Q. (They are extremely simple.)

C) Substitute the expressions for the time dependence of P and Q into Eqs. 29 and 30 to find the time dependence pf p and q.

Solution

A) We only need to verify that $[P, Q]$ equals 1. We have

$$[P, Q] = \frac{\partial P}{\partial p}\frac{\partial Q}{\partial q} - \frac{\partial P}{\partial q}\frac{\partial Q}{\partial p} = -(-mg)\frac{1}{mg} = 1 \tag{31}$$

so the transformation is canonical.

B) The new Hamiltonian is

$$H' = \frac{(mgQ)^2}{2m} + mg\Big(\frac{P}{mg} - \frac{gQ^2}{2}\Big) = P \tag{32}$$

The equations of motion are

$$\dot{P} = 0 \tag{33}$$

$$\dot{Q} = 1 \tag{34}$$

$P = \text{const}$ is the total energy E. The time dependence of Q is $Q(t) = t - t_0$ where t_0 is the time at which $Q = 0$.

C) Substituting the above results for the time evolution of P and Q into Eqs. 29, 30 we get

$$p = -mg(t - t_0) \tag{35}$$

$$q = \frac{E}{mg} - \frac{g(t - t_0)^2}{2} \tag{36}$$

where t_0 is the time at which the velocity of the object vanishes.

Problem 4

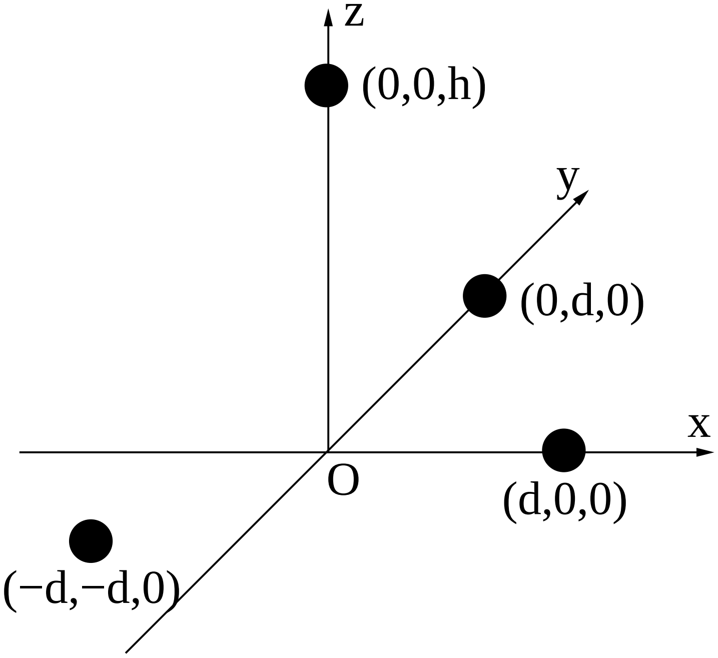

Figure 3: Illustration for problem 4.

Calculate the tensor of inertia with respect to the origin O of a system composed of four point-like objects of equal mass m located at $(d, 0, 0)$, $(0, d, 0)$, $(-d, -d, 0)$, and $(0, 0, h)$. Find then the three principal axes of inertia and the corresponding principal moments of inertia.

Solution

Denoting the coordinates of the four objects by $r_{i,k}$, $i = 1, 2, 3$ for x, y, z and $k = 1, 2, 3, 4$ the tensor of inertia will be given by

$$I = m \times \begin{pmatrix} \sum_k (r_{2,k}^2 + r_{3,k}^2) & -\sum_k r_{1,k} r_{2,k} & -\sum_k r_{1,k} r_{3,k} \\ -\sum_k r_{1,k} r_{2,k} & \sum_k (r_{1,k}^2 + r_{3,k}^2) & -\sum_k r_{2,k} r_{3,k} \\ -\sum_k r_{1,k} r_{3,k} & -\sum_k r_{2,k} r_{3,k} & \sum_k (r_{1,k}^2 + r_{2,k}^2) \end{pmatrix} \tag{37}$$

With $r_{1,1} = d, r_{2,2} = d, r_{1,3} = -d, r_{2,3} = -d, r_{3,4} = h$ and all other $r_{i,k} = 0$, we find

$$I = m \times \begin{pmatrix} 2d^2 + h^2 & -d^2 & 0 \\ -d^2 & 2d^2 + h^2 & 0 \\ 0 & 0 & 4d^2 \end{pmatrix} \tag{38}$$

The matrix is block diagonal and we only need to find eigenvectors and eigenvalues of the upper 2×2 block. Given the symmetry, the eigenvectors, and thus the first two principal axes of inertia, are

$$\hat{n}_1 = \left(\frac{\sqrt{2}}{2}, \frac{\sqrt{2}}{2}, 0\right) \tag{39}$$

$$\hat{n}_2 = \left(\frac{\sqrt{2}}{2}, -\frac{\sqrt{2}}{2}, 0\right) \tag{40}$$

while the third axis is the z axis. The corresponding principal moments of inertia are

$$I_1 = \hat{n}_1 \cdot I \hat{n}_1 = d^2 + h^2 \tag{41}$$

$$I_2 = \hat{n}_2 \cdot I \hat{n}_2 = 3d^2 + h^2 \tag{42}$$

$$I_2 = 4d^2 \tag{43}$$