PY 408 - Intermediate Mechanics

Assignment 9 - November 29, 2011.

Due in class on December 6, 2011.

Please note: barring truly exceptional situations, late assignments will not be accepted.

Do all four problems. Each correct solution will be given 25 points, with points taken away for errors or for lack of explanation, according to the severity of the error.

Problem 1

Consider the double pendulum as studied in class (see Lecture notes 3, page 2 and following pages), but now take the length and mass of the top pendulum be ℓ_1, m_1 and those of the bottom pendulum to be ℓ_2, m_2. Write down the equations for small oscillations about the equilibrium position $\theta_1 = \theta_2 = 0$ and find the frequencies of the two normal modes of oscillation (you do not have to calculate the corresponding amplitudes.) Check that in the limit $m_1 \gg m_2$ the frequencies of the normal modes reduce to those of the two individual pendulums.

Solution

We have

$$\begin{aligned} x_1 &= \ell_1 \sin\theta_1 \\ z_1 &= -\ell_1 \cos\theta_1 \\ x_2 &= \ell_1 \sin\theta_1 + \ell_2 \sin\theta_2 \\ z_2 &= -\ell_1 \cos\theta_1 + \ell_2 \cos\theta_2 \end{aligned} \tag{1}$$

and

$$\begin{aligned} \dot{x}_1 &= \ell_1 \dot{\theta}_1 \cos\theta_1 \\ \dot{z}_1 &= \ell_1 \dot{\theta}_1 \sin\theta_1 \\ \dot{x}_2 &= \ell_1 \dot{\theta}_1 \cos\theta_1 + \ell_2 \dot{\theta}_2 \cos\theta_2 \\ \dot{z}_2 &= \ell_1 \dot{\theta}_1 \sin\theta_1 + \ell_2 \dot{\theta}_2 \sin\theta_2 \end{aligned} \tag{2}$$

The kinetic energy is

$$K = \frac{m_1}{2}(\dot{x}_1^2 + \dot{z}_1^2) + \frac{m_2}{2}(\dot{x}_2^2 + \dot{z}_2^2) =$$
$$\frac{m_1 + m_2}{2}\ell_1^2\dot{\theta}_1^2 + m_2\ell_1\ell_2\dot{\theta}_1\dot{\theta}_2\cos(\theta_1 - \theta_2) + \frac{m_2}{2}\ell_2^2\dot{\theta}_2^2 \quad (3)$$

In the limit of small oscillations we need to keep only the terms quadratic in the velocities and K can be approximated by

$$K_2 = \frac{m_1 + m_2}{2}\ell_1^2\dot{\theta}_1^2 + m_2\ell_1\ell_2\dot{\theta}_1\dot{\theta}_2 + \frac{m_2}{2}\ell_2^2\dot{\theta}_2^2 \quad (4)$$

The potential energy is

$$V = mg(z_1 + z_2) = -(m_1 + m_2)g\ell_1\cos\theta_1 - m_2\ell_2\cos\theta_2 \quad (5)$$

In the limit of small oscillations we need to keep only the quadratic terms in the expansion of the cosines and V can be approximated by

$$V_2 = (m_1 + m_2)g\ell_1\frac{\theta_1^2}{2} + m_2g\ell_2\frac{\theta_2^2}{2} \quad (6)$$

The equations of motion which follow from the Lagrangian $L = K_2 - V_2$ are

$$\begin{aligned}(m_1 + m_2)\ell_1^2\ddot{\theta}_1 + m_2\ell_1\ell_2\ddot{\theta}_2 &= -(m_1 + m_2)g\ell_1\theta_1 \\ m_2\ell_1\ell_2\ddot{\theta}_1 + m_2\ell_2^2\ddot{\theta}_2 &= -m_2g\ell_2\theta_2 \quad (7)\end{aligned}$$

We seek solutions of the form

$$\begin{aligned}\theta_1(t) &= a_1e^{\imath\omega t} \\ \theta_2(t) &= a_2e^{\imath\omega t} \quad (8)\end{aligned}$$

Substituting and simplifying we get

$$\begin{aligned}[(m_1 + m_2)\ell_1^2\omega^2 - (m_1 + m_2)g\ell_1]a_1 &+ m_2\ell_1\ell_2\omega^2a_2 = 0 \\ m_2\ell_1\ell_2\omega^2a_1 &+ [m_2\ell_2^2\omega^2 - m_2g\ell_2]a_2 = 0 \quad (9)\end{aligned}$$

The equations will have a non-vanishing solution only if the determinant of the matrix of coefficients vanishes, which gives us the condition

$$[(m_1 + m_2)\ell_1^2\omega_2 - (m_1 + m_2)g\ell_1][m_2\ell_2^2\omega^2 - m_2g\ell_2] - m_2^2\ell_1^2\ell_2^2\omega^4 =$$
$$m_1m_2\ell_1^2\ell_2^2\omega^4 - (m_1 + m_2)m_2g\ell_1\ell_2(\ell_1 + \ell_2)\omega^2 + (m_1 + m_2)m_2g^2\ell_1\ell_2 = 0 \quad (10)$$

or, upon further simplification

$$m_1\ell_1\ell_2\omega^4 - (m_1+m_2)g(\ell_1+\ell_2)\omega^2 + (m_1+m_2)g^2 = 0 \tag{11}$$

The angular frequencies of the normal modes are then given by

$$\omega_{1,2}^2 = g\frac{(m_1+m_2)(\ell_1+\ell_2) \pm \sqrt{(m_1+m_2)[m_1(\ell_1-\ell_2)^2 + m_2(\ell_1+\ell_2)^2]}}{2m_1\ell_1\ell_2} \tag{12}$$

For $m_1 \gg m_2$ we may neglect m_2 and we get

$$\omega_{1,2}^2 = g\frac{m_1(\ell_1+\ell_2) \pm \sqrt{m_1^2(\ell_1-\ell_2)^2}}{2m_1\ell_1\ell_2} \tag{13}$$

or

$$\begin{aligned} \omega_1 &= \frac{g}{\ell_1} \\ \omega_2 &= \frac{g}{\ell_2} \end{aligned} \tag{14}$$

which are the (square) angular frequencies of the two decoupled pendulums.

Problem 2

A) Solve the equation for a damped oscillator

$$m\ddot{q} + b\dot{q} + kq = 0 \tag{15}$$

with initial conditions:

i) $q(t=0) = 1, \quad \dot{q}(t=0) = 0$

ii) $q(t=0) = 0, \quad \dot{q}(t=0) = 1$

B) Write the formulas for the solutions i) and ii) first by taking $m = 0.5, b = 1, k = 5$ and then by taking $m = 0.5, b = 3, k = 2.5$. Then plot the four solutions. (If you can generate a plot and print it, attach the plots to your homework. If not use a calculator to graph the solutions and draw the plots to the best of your ability.)

C) Find the most general solution to Eq. 15 in the case when $b = 2\sqrt{mk}$.

Solution

A) We look for solutions of the form

$$q(t) = e^{-\alpha t} \tag{16}$$

Substituting into Eq. 15 and simplifying we get the condition for α

$$m\alpha^2 - b\alpha + kq = 0 \tag{17}$$

with solutions

$$\alpha_\pm = \frac{b \pm \sqrt{b^2 - 4mk}}{2m} \tag{18}$$

The most general solutions will be a linear combination

$$q(t) = c_+ e^{-\alpha_+ t} + c_- e^{-\alpha_- t} \tag{19}$$

We have

$$q(0) = c_+ + c_- \tag{20}$$

$$\dot{q}(0) = -(c_+\alpha_+ + c_-\alpha_-) \tag{21}$$

i) Demanding $q(0) = 1,\ \dot{q}(0) = 0$ gives the conditions

$$c_+ + c_- = 1 \tag{22}$$

$$(c_+\alpha_+ + c_-\alpha_-) = 0 \tag{23}$$

which are solved by

$$\begin{aligned} c_+ &= \frac{\alpha_-}{\alpha_- - \alpha_+} \\ c_- &= \frac{\alpha_+}{\alpha_+ - \alpha_-} \end{aligned} \tag{24}$$

ii) Demanding $q(0) = 0,\ \dot{q}(0) = 1$ gives the conditions

$$c_+ + c_- = 0 \tag{25}$$

$$(c_+\alpha_+ + c_-\alpha_-) = -1 \tag{26}$$

which are solved by

$$\begin{aligned} c_+ &= \frac{1}{\alpha_- - \alpha_+} \\ c_- &= \frac{1}{\alpha_+ - \alpha_-} \end{aligned} \tag{27}$$

B) With $m = 0.5, b = 1, k = 5$ we have $\alpha_\pm = 1 \pm 3\imath$ and the two solutions correspond to

$$\begin{aligned} c_+ &= \frac{1-3\imath}{-6\imath} = \frac{1}{2} + \frac{\imath}{6} \\ c_- &= \frac{1+3\imath}{6\imath} = \frac{1}{2} - \frac{\imath}{6} \end{aligned} \tag{28}$$

and

$$\begin{aligned} c_+ &= \frac{1}{-6\imath} = \frac{\imath}{6} \\ c_- &= \frac{1}{6\imath} = \frac{-\imath}{6} \end{aligned} \tag{29}$$

The full solutions are given by

$$\frac{3\cos(3t) + \sin(3t)}{3} e^{-t} \tag{30}$$

and

$$\frac{\sin(3t)}{3} e^{-t} \tag{31}$$

With $m = 0.5, b = 3, k = 2.5$ we have $\alpha_\pm = 3 \pm 2$, or $\alpha_+ = 5$, $\alpha_- = 1$ and the two solutions correspond to

$$\begin{aligned} c_+ &= -\frac{1}{4} \\ c_- &= \frac{5}{4} \end{aligned} \tag{32}$$

and

$$\begin{aligned} c_+ &= -\frac{1}{4} \\ c_- &= \frac{1}{4} \end{aligned} \tag{33}$$

The full solutions are given by

$$\frac{-e^{-5t} + 5e^{-t}}{4} \tag{34}$$

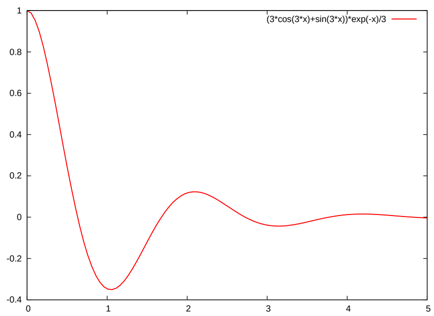

Figure 1: Solution with $m = 0.5, b = 1, k = 5$ and $\dot{q}(0) = 0$.

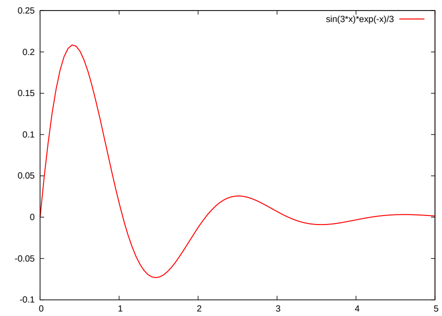

Figure 2: Solution with $m = 0.5, b = 1, k = 5$ and $q(0) = 0$.

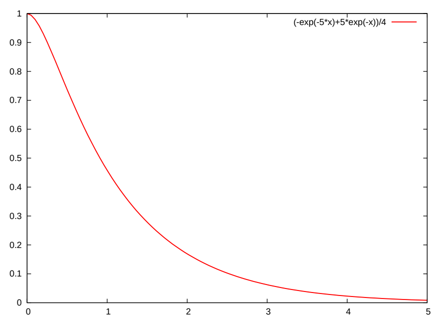

Figure 3: Solution with $m = 0.5, b = 3, k = 2.5$ and $\dot{q}(0) = 0$.

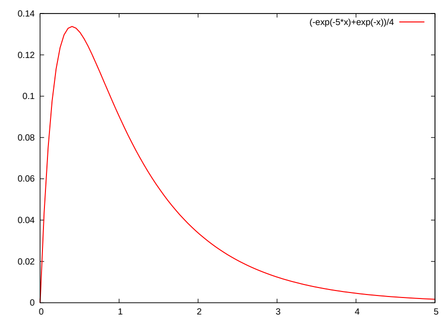

Figure 4: Solution with $m = 0.5, b = 3, k = 2.5$ and $q(0) = 0$.

and

$$\frac{-e^{-5t} + e^{-t}}{4} \tag{35}$$

All the solutions are plotted in the figures.

C) This is tricky: when $b = 2\sqrt{mk}$ the two solutions $\exp(-\alpha_\pm t)$ coincide. We can obtain another, linearly independent solution by a limiting procedure. If b differs from $2\sqrt{mk}$ by some infinitesimal amount db, α_+ and α_- will also differ by an infinitesimal amount $d\alpha$. For definiteness let us take $\alpha_+ = \alpha_- + d\alpha$. Let us now consider the linear combination of the two solutions

$$q(t) = -\frac{e^{-\alpha_+ t} - e^{-\alpha_- t}}{d\alpha} = -e^{-\alpha_- t}\frac{e^{-d\alpha\, t} - 1}{d\alpha} \tag{36}$$

In the limit $d\alpha \to 0$, $\alpha_+, \alpha_- \to \alpha$ this tends to

$$q(t) = te^{-\alpha t} \tag{37}$$

which is the other solution we were looking for. It is easy to check directly that $q(t) = t\exp(-\alpha t)$ satisfies the equation. Indeed we have $\dot{q}(t) = (1 - \alpha t)\exp(-\alpha t)$, $\ddot{q}(t) = (-2\alpha + \alpha^2 t)\exp(-\alpha t)$ and

$$m\ddot{q} + b\dot{q} + kq = (-2m\alpha + b)e^{-\alpha t} + (m\alpha^2 + b\alpha + k)te^{-\alpha t} \tag{38}$$

The second term in the r.h.s. vanishes by virtue of Eq. 17, while the first term vanishes because with $b = 2\sqrt{mk}$, $\alpha = b/2m$.

Problem 3

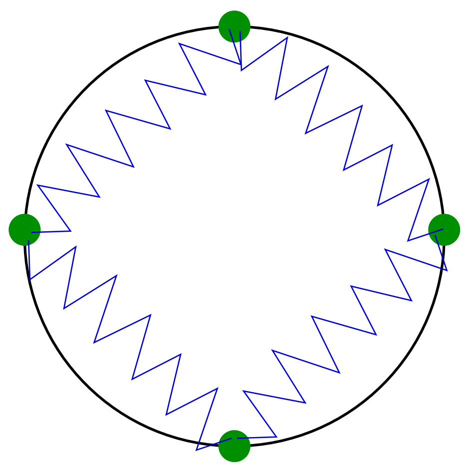

Figure 5: Illustration for problem 3.

A) Consider a system made of 4 point-like objects which are constrained to move on a circumference of radius R in the $x - y$ plane, as shown in Fig. 5. (The value of R is actually irrelevant for this problem. It will only serve for illustration purposes.) The four objects have equal mass m and are connected by four equal springs, as illustrated in the figure. Let us denote by q_j, with $j = 0 \div 3$, small displacements along the circumference of the bodies from the equilibrium positions $x_0 = 0, y_0 = R$; $x_1 = R, y_1 = 0$; $x_2 = 0, y_2 = -R$;

$x_3 = -R, y_3 = 0$. The small oscillation Lagrangian is given by

$$L = \frac{m(\dot{q}_0^2 + \dot{q}_1^2 + \dot{q}_2^2 + \dot{q}_3^2)}{2}$$
$$-\frac{k[(q_0 - q_1)^2 + (q_1 - q_2)^2 + (q_2 - q_3)^2 + (q_3 - q_0)^2]}{2} \tag{39}$$

Write down the equations of motion that follow from this Lagrangian and find the normal modes of oscillation (both the angular frequencies and the amplitudes.)

B) Generalize the above problem to the case of N objects around the circumference with a small oscillation Lagrangian

$$L = \frac{m \sum_{j=0}^{N-1} \dot{q}_j^2}{2} - \frac{k \sum_{j=0}^{N-1} (q_j - q_{j+1})^2}{2} \tag{40}$$

where the last sum must be taken with periodic boundary conditions, i.e. identifying the index N with 0, or, more explicitly, by interpreting

$$\sum_{j=0}^{N-1} (q_j - q_{j+1})^2 \quad \text{as} \quad \sum_{j=0}^{N-2} (q_j - q_{j+1})^2 + (q_{N-1} - q_0)^2 \tag{41}$$

Write down again the equations of motion that follow from this Lagrangian and find the normal modes of oscillation (both the angular frequencies and the amplitudes.)

Solution

We solve A) and B) together. The equations of motion which follow from the Lagrangian in Eq. 40 are

$$m\ddot{q}_j = k(q_{j+1} + q_{j-1} - 2q_j) \tag{42}$$

Prompted by the symmetry under cyclic permutation ($j \to j+1$) we look for solutions of the form

$$q_j(t) = e^{2\pi \imath j n/N} e^{\imath \omega t} \tag{43}$$

with integer n. Substituting we get

$$-m\omega^2 e^{2\pi \imath j n/N} e^{\imath \omega t} = k\left(e^{2\pi \imath (j+1) n/N} e^{\imath \omega t} + e^{2\pi \imath (j-1) n/N} e^{\imath \omega t} - 2 e^{2\pi \imath j n/N} e^{\imath \omega t}\right) \tag{44}$$

or, eliminating the common factor $-\exp(2\pi\imath jn/N)\times\exp(\imath\omega t)$

$$m\omega^2 = k\big(2 - e^{2\imath\pi n/N} - e^{-2\imath\pi n/N}\big) = k[2 - 2\cos(2\pi n/N)] \tag{45}$$

Thus we see that we will have a solution, i.e. a normal mode of oscillation, for each value of n from 0 through $N-1$ (values of n outside this range just reproduce one of the solutions with n in the range $0 \div N-1$.) The corresponding angular frequencies are

$$\omega^n_{\pm} = \pm\sqrt{\frac{k[2 - 2\cos(2\pi n/N)]}{m}} \tag{46}$$

and the amplitudes, normalized to 1, are given by the vectors

$$v^{(n)}_j = \frac{1}{\sqrt{N}} e^{2\pi\imath jn/N} \tag{47}$$

Note the existence of a zero mode solution (a normal mode with $\omega = 0$) which reflects the degeneracy of the equilibrium configurations: any common rotation of the chosen equilibrium positions would lead to another equilibrium position.

For $N = 4$ we get the frequencies $\omega = 0, \sqrt{2k/m}$ twice, $2\sqrt{k/m}$, with amplitudes $(1/2, 1/2, 1/2, 1/2)$, $(1/2, \imath/2, -1/2, -\imath/2)$, $(1/2, -1/2, 1/2, -1/2)$, $(1/2, -\imath/2, -1/2, \imath/2)$.

Problem 4

Consider a rigid body formed by six point-like objects located at the vertices of a regular octahedron: precisely the system consists of two point-like objects of mass m_1 located at the points of coordinates $(d, 0, 0)$ and $(-d, 0, 0)$ of a Cartesian reference system, two point-like objects of mass m_2 located at $(0, d, 0)$ and $(0, -d, 0)$, and two point-like objects of mass m_3 located at $(0, 0, d)$ and $(0, 0, -d)$.

A) Calculate the inertia tensor of this system.

B) Find the values that m_1, m_2, m_3 should have so that the three principal moments of inertia take value I_1, I_2, I_3.

Solution

By symmetry the off-diagonal component of the tensor of inertia vanish, and the diagonal components are

$$\begin{aligned} I_{1,1} &= 2d^2(m_2 + m_3) \\ I_{2,2} &= 2d^2(m_1 + m_3) \\ I_{3,3} &= 2d^2(m_1 + m_2) \end{aligned} \tag{48}$$

Demanding that the three principal moments of inertia are equal to I_1, I_2, I_3 gives the equations

$$\begin{aligned} m_2 + m_3 &= \frac{I_1}{2d^2} \\ m_1 + m_3 &= \frac{I_2}{2d^2} \\ m_1 + m_2 &= \frac{I_3}{2d^2} \end{aligned} \tag{49}$$

Adding the three equations we get

$$m_1 + m_2 + m_3 = \frac{I_1 + I_2 + I_3}{4d^2} \tag{50}$$

from which we easily obtain

$$\begin{aligned} m_1 &= \frac{-I_1 + I_2 + I_3}{4d^2} \\ m_2 &= \frac{I_1 - I_2 + I_3}{4d^2} \\ m_3 &= \frac{I_1 + I_2 - I_3}{4d^2} \end{aligned} \tag{51}$$