Events. Space-time coordinates.

The points of space-time are "events": things that happen instantaneously at a certain space point and at a definite of time. We introduce a reference frame $\mathcal{R}$ in space-time by introducing an $(x, y, z)$ Cartesian coordinate frame in space and clocks that measure the time at all spatial points. Although we are used to think about time as universal, it will be important that time can be measured locally.

As an example of event, let us imagine that a nucleus located at $(x_A, y_A, z_A)$ decays at time $t_A$ emitting an electron (a $\beta$-ray). This is event A. The electron is detected by a detector located at $(x_B, y_B, z_B)$ at time $t_B$. This is event B.

We will label all events by their space-time coordinates, but it will be convenient to use along the time axis a coordinate which has dimension of length, like those along the $x, y, z$-axes. For this we will use $ct$, $c$ being the speed of light, as the coordinate along the time axis.

Very important will be events where a light ray is emitted at $(ct_A, x_A, y_A, z_A)$ (event A) and detected at $(ct_B, x_B, y_B, z_B)$ (event B). In between the two events the light covers a length

$$\ell = \sqrt{(x_B - x_A)^2 + (y_B - y_A)^2 + (z_B - z_A)^2}$$

in a time

$$t = t_B - t_A$$

Since the light moves with speed $c$ we will have

$$\ell = ct$$

It follows, rather trivially, that the quantity

$$d_{AB} = (ct_B - ct_A)^2 - (x_B - x_A)^2 - (y_B - y_A)^2 - (z_B - z_A)^2$$
is equal to zero. We will called the quantity in the r.h.s. of Eq. 4 the “squared space-time distance”, or simply “squared distance”, between events A and B, although, as opposed to the square of the distance between two space points which is always positive, the squared space-time distance between two events can be positive, zero, or negative. In particular, we have just seen that the squared space-time distance between the two events given by the emission and detection of light is always zero.

Change of reference frame.

We will consider now the relation between space-time coordinates in two reference frames $\mathcal{R}$ and $\mathcal{R}'$ which move with uniform velocity with respect to each other. For concreteness, and without loss of generality, we will assume that their space-time origin coincide, i.e. that an event which occurs at space position $(x = 0, y = 0, z = 0)$ at time $t = 0$ in $\mathcal{R}$ is also seen to occur at $(x' = 0, y' = 0, z' = 0)$ at time $t' = 0$ in $\mathcal{R}'$. Also without loss of generality we will assume that the relative motions of the two frames occur along their respective $x$ and $x'$ axes so that, for the moment, we can restrict our attention to the relation between the $ct, x$ and $ct', x'$ space-time coordinates in the two frames. The basic postulate of the theory of relativity can now be formulated as follows: If the squared-distance $d_{AB}$ between two events A and B is zero in frame $\mathcal{R}$, the corresponding squared-distance $d'_{AB}$ will also be zero in frame $\mathcal{R}'$.

The relation between $(ct, x)$ and $(ct', x')$ will be linear, otherwise uniform motion, i.e. motion with uniform velocity, in one of the two frames will not be uniform motion in the other. With our assumption that $ct = x = 0$ is mapped into $ct' = x' = 0$, the linear relation will also be homogeneous. Thus relation will be of the form

$$\begin{pmatrix} ct' \\ x' \end{pmatrix} = M \begin{pmatrix} ct \\ x \end{pmatrix}$$

(5)

where $M$ is a two by two matrix which we must determine. It turns out that the fact that the speed of light must be the same in the two frames, i.e. that $d_{A,B} = 0$ implies $d'_{A,B} = 0$ and vice versa, is sufficient to fix the form of $M$ apart from a multiplicative constant, but we can fix the form of $M$ completely by demanding that the space-time squared distance is left invariant, i.e. demanding that $d'_{A,B} = d_{A,B}$, even when $d$ is not zero. Because of the linearity of the transformation it will be sufficient to demand that
\(d'_{A,O} = d_{A,O}\) where \(O\) is the common origin of the two frames, i.e. to demand that

\[(ct')^2 - x'^2 = (ct)^2 - x^2\]  

(6)

We can get a clue to the solution of this requirement by considering the transformation of the coordinates \((x, y)\) of the ordinary plane when the frame of reference is rotated around the origin. The transformation is also a matrix transformation of the form

\[
\begin{pmatrix}
  x' \\
  y'
\end{pmatrix} = R
\begin{pmatrix}
  x \\
  y
\end{pmatrix}
\]  

(7)

and the form of the rotation matrix \(R\) is fully determined by the requirement that the squared distance from the origin, \(r^2 = x^2 + y^2\), is left invariant by the transformation. If one demands that

\[x^2 + y'^2 = x'^2 + y'^2\]  

(8)

it is easy to prove that the form of \(R\) must be

\[R(\phi) = \begin{pmatrix}
  \cos \phi & \sin \phi \\
  -\sin \phi & \cos \phi
\end{pmatrix}\]  

(9)

i.e. that the transformation is

\[
\begin{pmatrix}
  x' \\
  y'
\end{pmatrix} = \begin{pmatrix}
  \cos \phi & \sin \phi \\
  -\sin \phi & \cos \phi
\end{pmatrix}
\begin{pmatrix}
  x \\
  y
\end{pmatrix}
\]  

(10)

where \(\phi\) is an angle, which turns out to be that angle of rotation between the two frames\(^1\). Inspired by the fact that the identity \(\cos^2 \phi + \sin^2 \phi = 1\) plays a crucial role in proving that \(x' + y'^2 = x^2 + y^2\) and by the fact that the functions \(\cosh\) and \(\sinh\) satisfy a similar identity with the minus sign: \(\cosh^2 \eta - \sinh^2 \eta = 1\), and after some trials for the signs, we find that the solution for the matrix \(M\) of Eq. 5 is

\[M(\eta) = \begin{pmatrix}
  \cosh \eta & -\sinh \eta \\
  -\sinh \eta & \cosh \eta
\end{pmatrix}\]  

(11)

so that the relation between the space-time coordinates in the two frames will be

\[
\begin{pmatrix}
  ct' \\
  x'
\end{pmatrix} = \begin{pmatrix}
  \cosh \eta & -\sinh \eta \\
  -\sinh \eta & \cosh \eta
\end{pmatrix}
\begin{pmatrix}
  ct \\
  x
\end{pmatrix}
\]  

(12)

\(^1\)Beyond the invariance of \(r^2\) one must actually also demand that the orientation of the two planes is unchanged, but this is not important here.
or, explicitly,
\[
ct' = ct \cosh \eta - x \sinh \eta \\
x' = -ct \sinh \eta + x \cosh \eta
\]  

(13)
\[
x' = -ct \sinh \eta + x \cosh \eta
\]  

(14)

It is easy to prove that the transformation of Eq. 12 leaves the squared space-time distance unchanged, so that, in particular, the speed of light will be the same in the two frames. Indeed we have
\[
(ct')^2 - x'^2 = (ct \cosh \eta - x \sinh \eta)^2 - (-ct \sinh \eta + x \cosh \eta)^2
\]
\[
= (ct)^2(cosh^2 \eta - sinh^2 \eta) + 2(ct)x(-cosh \eta sinh \eta + sinh \eta cosh \eta)
\]
\[
+ x^2(sinh^2 \eta - cosh^2 \eta) = (ct)^2 - x^2
\]  

(15)

We may now extend our considerations to the full three-dimensional space. The \(y\) and \(z\) coordinates are left unchanged, so the full transformation in the change of reference system is given by
\[
\begin{pmatrix}
ct' \\
x' \\
y' \\
z'
\end{pmatrix} = \begin{pmatrix}
cosh \eta & -\sinh \eta & 0 & 0 \\
-\sinh \eta & \cosh \eta & 0 & 0 \\
0 & 0 & 1 & 0 \\
0 & 0 & 0 & 1
\end{pmatrix} \begin{pmatrix}
ct \\
x \\
y \\
z
\end{pmatrix}
\]  

(16)

We will still have \(d'_AB = d_{AB}\) for all events or
\[
(ct'_B - ct'_A)^2 - (x'_B - x'_A)^2 = (ct_B - ct_A)^2 - (x_B - x_A)^2
\]
\[
= (ct_B - ct_A)^2 - (x_B - x_A)^2 - (y_B - y_A)^2 - (z_B - z_A)^2
\]  

(17)

since, as we have seen above, \((ct'_B - ct'_A)^2 - (x'_B - x'_A)^2 = (ct_B - ct_A)^2 - (x_B - x_A)^2\) for all separations (here we see why it is important that Eq. 15 holds for all values of the coordinates, not just when their squared distance from the origin is zero) and, obviously, \((y_B - y_A)^2 - (z_B - z_A)^2 = (y_B - y_A)^2 - (z_B - z_A)^2\).

The transformation of coordinates of Eq. 16 is called the "Lorentz transformation" and the parameter \(\eta\) is called "rapidity".

The Lorentz transformation.

We make contact here with the more conventional expression for the Lorentz transformation, the one more likely to be found in textbooks.
Consider first the motion of the $\mathcal{R}'$ frame as seen by an observer in $\mathcal{R}$. The origin $x' = 0$ of the $x'$ axis of $\mathcal{R}'$ is located at $x = 0$ for $t = 0$, since we assumed that the origins of the space axes in the two frames overlap at time equal zero in both frames (event $O : (x = 0, t = 0), (x' = 0, t' = 0)$.) As $t$ increases from 0 the origin $x' = 0$ of $\mathcal{R}'$ moves along the $x$-axis of $\mathcal{R}$ as we can see from Eq. 14: $x' = -ct \sinh \eta + x \cosh \eta$, which, setting $x' = 0$, gives

$$0 = -ct \sinh \eta + x \cosh \eta$$  \hspace{1cm} (18)

or

$$x = ct \frac{\sinh \eta}{\cosh \eta} = (c \tanh \eta)t$$  \hspace{1cm} (19)

We see from this equation that in $\mathcal{R}$ the $\mathcal{R}'$ frame moves in the $x$ direction with velocity

$$v = c \tanh \eta$$  \hspace{1cm} (20)

(For an observer in the $\mathcal{R}'$ frame the $\mathcal{R}$ frame moves along the $x'$-axis with velocity $-v$. It is easy to check that the inverse of $M(\eta)$ is $M(-\eta)$.) In textbooks the ratio $v/c$ is generally denoted by $\beta$. We see then that

$$\beta = \frac{v}{c} = \tanh \eta$$  \hspace{1cm} (21)

Then, from

$$\cosh^2 \eta - \sinh^2 \eta = 1$$  \hspace{1cm} (22)

we find first

$$1 - \frac{\sinh^2 \eta}{\cosh^2 \eta} = 1 - \tanh^2 \eta = \frac{1}{\cosh^2 \eta}$$  \hspace{1cm} (23)

and then

$$\cosh \eta = \frac{1}{\sqrt{1 - \tanh^2 \eta}} = \frac{1}{\sqrt{1 - \beta^2}}$$  \hspace{1cm} (24)

In textbooks the quantity at the r.h.s. of this equation is generally denoted by $\gamma$:

$$\gamma = \frac{1}{\sqrt{1 - \beta^2}}$$  \hspace{1cm} (25)

So we have

$$\cosh \eta = \gamma$$  \hspace{1cm} (26)

and

$$\sinh \eta = \tanh \eta \cosh \eta = \beta \gamma$$  \hspace{1cm} (27)
With these identities the Lorentz transformation takes the form

\[ ct' = \gamma ct - \beta \gamma x \]  
\[ x' = -\beta \gamma ct + \gamma x \]  

or

\[ t' = \gamma \left( t - \frac{v}{c^2}x \right) \]  
\[ x' = \gamma (x - vt) \]

which are the expressions one is more likely to find in textbooks.

We notice that \( \beta \) can vary between -1 and 1: \(-1 < \beta < 1\), the end-points being approached as \( v \to c \). \( \gamma \) is never smaller than 1: \( 1 \leq \gamma < \infty \) with \( \gamma \to \infty \) as \( v \to c \). This agrees of course with our knowledge of the behavior of \( \tanh \eta \) and \( \cosh \eta \). We also note that for small \( v \)

\[ \gamma(v) = \left[ 1 - \left( \frac{v}{c} \right)^2 \right]^{-1/2} \approx 1 + \frac{v^2}{2c^2} \]  

Thus we see that to the extent that \( v \) is much smaller than \( c \), so that the terms with \( c^2 \) at the denominator can be neglected, the Lorentz transformation reduces to

\[ t' = t \]  
\[ x' = x - vt \]  

i.e. time is universal and the relation between \( x \) and \( x' \) is the Newtonian (or Galilean) relation between inertial frames.

**Why the rapidity formulation?**

There are advantages in being familiar with the rapidity formulation of the Lorentz transformation as well as the one based on the parameters \( \beta \) and \( \gamma \).

At a more theoretical level, the rapidity formulation puts in evidence the relation between the transformations in ordinary space, which preserve the squared distance

\[ d_{AB} = (x_B - x_A)^2 + (y_B - y_A)^2 + (z_B - z_A)^2 \]  

\[ 6 \]
and those in space-time, which preserve the squared distance

\[ d_{AB} = (ct_B - ct_A)^2 - (x_B - x_A)^2 - (y_B - y_A)^2 - (z_B - z_A)^2 \]  \hspace{1cm} (36)

In geometry the spaces, in any number of dimension, where the squared distance in given by Eq. 35 (generalized of course according to the number of dimensions) are called “Euclidean”, and those where the squared distance has the form of Eq. 36 are called “Minkowskian”, or “Minkowski spaces” (from the name of the German mathematician Hermann Minkowski.) Lorentz transformations can thus be seen as the equivalent in Minkowski space of rotations in Euclidean spaces. We can push this analogy a little further if we are willing to go to complex variables. Let us limit, for simplicity, our attention to spaces of dimensionality two, and consider first the Euclidean space with coordinates \( x, y \) and squared distance from the origin

\[ d = x^2 + y^2 \]  \hspace{1cm} (37)

Let us now make the replacement \( y \to iu \). \( d \) will take the form

\[ d = x^2 + (iu)^2 = x^2 + i^2u^2 = x^2 - u^2 \]  \hspace{1cm} (38)

which has the characteristic form of the squared distance in two-dimensional Minkowski space. Let us then consider the Euclidean rotation

\[ x' = x \cos \phi + y \sin \phi \]  \hspace{1cm} (39)
\[ y' = -x \sin \phi + y \cos \phi \]  \hspace{1cm} (40)

and replace \( y \) with \( iu \). We get

\[ x' = x \cos \phi + iu \sin \phi \]  \hspace{1cm} (41)
\[ uu' = -x \sin \phi + iu \cos \phi \]  \hspace{1cm} (42)

This transformation is seen to mix real and imaginary numbers, but we can get rid of the inconsistency by letting \( \phi \) become imaginary as well. So we make the replacement \( \phi \to i\eta \) and use the relations \( \cos \eta = \cosh \eta \), \( \sin \eta = i \sinh \eta \) to obtain

\[ x' = x \cos \eta + uu \sin \eta = x \cosh \eta + (i)^2u \sinh \eta \]  \hspace{1cm} (43)
\[ uu' = -x \sin \eta + uu \cos \eta = -ix \sinh \eta + uu \cosh \eta \]  \hspace{1cm} (44)
or, after simplification of the common factor \( \nu \) in Eq. 44,

\[
\begin{align*}
    x' &= x \cosh \eta - u \sinh \eta \\
    u' &= -x \sinh \eta + u \cosh \eta
\end{align*}
\] (45) (46)

With the obvious identifications \( x \to ct, u \to x \) this transformation is seen to be the Lorentz transformation of Eqs. 13, 14.

On a less conceptual level, let us see what happens if the change of frame from \( \mathcal{R} \) to \( \mathcal{R}' \) moving in \( \mathcal{R} \) with rapidity \( \eta \) (i.e. with velocity \( v = c \tanh \eta \)) is followed by a change of frame from \( \mathcal{R}' \) to a frame \( \mathcal{R}'' \) moving in \( \mathcal{R}' \) with rapidity \( \eta' \) (i.e. with velocity \( v' = c \tanh \eta' \)) with respect to \( \mathcal{R}' \). A change of frame to a frame moving with rapidity \( \eta \), or velocity \( v \), is often referred to as a “boost”. So, the question we are addressing here is what happens if a boost of rapidity \( \eta \) is followed by a boost of rapidity \( \eta' \). We consider only the \( ct \) and \( x \) axes, since the Lorentz transformation of the transverse coordinates \( y, z \) is trivial. After the first boost we get

\[
\begin{pmatrix}
    ct' \\
    x'
\end{pmatrix} = M(\eta) \begin{pmatrix}
    ct \\
    x
\end{pmatrix}
\] (47)

With the further boost of rapidity \( \eta' \) in \( \mathcal{R}' \) we will have

\[
\begin{pmatrix}
    ct'' \\
    x''
\end{pmatrix} = M(\eta') \begin{pmatrix}
    ct' \\
    x'
\end{pmatrix}
\] (48)

Combining the two transformations we obtain

\[
\begin{pmatrix}
    ct'' \\
    x''
\end{pmatrix} = M(\eta')M(\eta) \begin{pmatrix}
    ct \\
    x
\end{pmatrix}
\] (49)

By performing the matrix multiplication we find

\[
M(\eta')M(\eta) = \begin{pmatrix}
    \cosh \eta' & -\sinh \eta' \\
    -\sinh \eta' & \cosh \eta'
\end{pmatrix} \begin{pmatrix}
    \cosh \eta & -\sinh \eta \\
    -\sinh \eta & \cosh \eta
\end{pmatrix} = \begin{pmatrix}
    \cosh(\eta + \eta') & -\sinh(\eta + \eta') \\
    -\sinh(\eta + \eta') & \cosh(\eta + \eta')
\end{pmatrix}
\] (50)

where we have used the addition formulae for hyperbolic functions. Thus we see that a boost by \( \eta \) in \( \mathcal{R} \) followed by a boost by \( \eta' \) in \( \mathcal{R}' \) is equivalent to
a boost by $\eta + \eta'$ in the original frame $\mathcal{R}$. Notice the analogy with what happens with rotations in the $x - y$ plane where a rotation of frame by an angle $\phi$ followed by a rotation of the new frame by an angle $\phi'$ is equivalent to a rotation of the original frame by an angle $\phi + \phi'$.

If we were to do the same calculation with the Lorentz transformations expressed in terms of $\beta$ and $\gamma$ we would run into a more involved algebra. But we can anticipate the outcome. As we have just seen, the combined boosts are equivalent to a single boost in the original frame with rapidity $\eta + \eta'$, that is to a boost with velocity

$$v'' = c \tanh(\eta + \eta')$$  \hspace{1cm} (51)

Now we have

$$\tanh(\eta + \eta') = \frac{\sinh(\eta + \eta')}{\cosh(\eta + \eta')} = \frac{\sinh \eta \cosh \eta' + \cosh \eta \sinh \eta'}{\cosh \eta \cosh \eta' + \sinh \eta \sinh \eta'}$$  \hspace{1cm} (52)

or, by dividing numerator and denominator by $\cosh \eta \cosh \eta'$,

$$\tanh(\eta + \eta') = \frac{[\sinh \eta / \cosh \eta] + [\sinh \eta'/ \cosh \eta']} {1 + [\sinh \eta / \cosh \eta][\sinh \eta'/ \cosh \eta']}
\hspace{1cm} = \frac{\tanh \eta + \tanh \eta'} {1 + \tanh \eta \tanh \eta'} = \frac{v/c + v'/c} {1 + vv'/c^2}$$  \hspace{1cm} (53)

Thus we finally find

$$v'' = \frac{v + v'} {1 + vv'/c^2}$$  \hspace{1cm} (54)

One can prove that, so long as $v$ and $v'$ have magnitude smaller than $c$, $v''$ will also have magnitude smaller than $c$. But we knew that already, since $v'' = c \tanh(\eta + \eta')$ and the magnitude of the hyperbolic tangent function is never larger than 1.

**Time dilation and space contraction.**

For the considerations that follow it will be convenient to refer to observers in frame $\mathcal{R}$ as “we”, or “us”, and to frame $\mathcal{R}'$ as the frame of a spaceship, which moves with velocity $v$ with respect to us. Let us imagine that the spaceship has length $\ell$ (in its own frame) extending from $x' = 0$ to $x' = \ell$. 
Time dilation.

Let us further imagine that at the end of the spaceship, i.e. at $x' = 0$ there is a beacon that sends out a light signal at regular intervals of duration $T$ in the spaceship frame. The emissions of the light signal are events which we denote by $A_0, A_1 \ldots A_n$. They all have coordinate $x' = 0$ of course and, assuming that the first signal is emitted at $t' = 0$, the events will have coordinates $A_0 : (0,0); A_1 : (cT,0); \ldots A_n : (ncT,0)$ in the spaceship frame $\mathcal{R}'$. We would like to find the coordinates of the events $A_n$ in our own frame. For this we need to invert the Lorentz transformation of Eqs. 28, 29 and to express $ct, x$ in terms of $ct', x'$. This is easy, because we just need to replace $v$ with $-v$ (i.e. $\beta$ with $-\beta$), obtaining

$$
ct = \gamma ct' + \beta \gamma x' \tag{55}
$$

$$
x = \beta \gamma ct' + \gamma x' \tag{56}
$$

With $x' = 0$ and $t' = nT$, Eq. 55 gives us

$$
t_n = \gamma nT = n \frac{T}{\sqrt{1 - v^2/c^2}} \tag{57}
$$

that is, the interval between two subsequent flashes of the beacon is longer in our frame than it is in the spaceship frame. This is the phenomenon of “relativistic time dilation”. It is most noticeable in the world of subatomic particles. For example the $\mu$ mesons, or muons, have an average lifetime of approx. 1 $\mu$s (one microsecond = $10^{-6}$s). In the experiments to measure their magnetic moment with very high precision muons are accelerated to a velocity very close to $c$ and kept by a magnetic field within a large ring, where they go through many revolutions while signals due to their magnetic moment are measured. Now in one microsecond light travels for $3 \times 10^8 \text{m/s} \times 10^{-6}\text{s} = 300\text{m}$ which is much less than the distance the muons travel in their many turns around the ring before decaying. The explanation is that, because of time dilation, the life span of one microsecond in the muon’s frame becomes a much longer time in our own frame.

It is interesting to observe that Eq. 56 for the $x$ coordinate of the events $A_n$ (the $x$ coordinate in our frame) is consistent with the fact that the spaceship moves with velocity $v$ in our frame. Indeed Eq. 56 gives us

$$
x_n = \beta \gamma ct'_n = \nu \gamma nT = \nu t_n \tag{58}
$$
as it should be.

It is also important to observe that the times $t_n$ are the local times, in our frame, of the events $A_n$, i.e. the time measured by the clock in our frame that happens to be at the position where the light signal is emitted by the flashing beacon. If we sit at our origin, i.e. at $x = 0$, we will see the flashes of light when they reach our eyes: let us denote these events $B_n$, with coordinates (in our frame) $\tilde{t}_n$, and $\tilde{x}_n = 0$. We can find the value of $\tilde{t}_n$ from the constraint that the squared separation between $A_n$ and $B_n$ is zero, because it is the separation between emission and detection of light: $c^2(\tilde{t}_n - t_n)^2 - x_n^2 = 0$. The solution of this equation, demanding also that $\tilde{t}_n > t_n$, is

$$\tilde{t}_n = t_n + \frac{x_n}{c} = \gamma n T \left( 1 + \frac{v}{c} \right) = \sqrt{\frac{1 + v/c}{1 - v/c}} nT$$  \hspace{1cm} (59)$$

It is an amusing exercise to imagine that $T$ is, for example, equal to one year, and that after 5 years in its own time the spaceship reverses course. In 5 more of its own years the spaceship will get back to the origin of our $x$-axis, the place it left from. The exercise consists in listing all the times (our times) at which we received the light signals from the spaceship’s beacon. The last signal, emitted at event $A_{10}$, will be received instantly because the spaceship will be back at $x = 0$, and our clock will show $\gamma(v) \times 10$ years.

**Length contraction.**

Let us imagine that as the spaceship travels we measure its length at the moment when its beacon, which is located at $x' = 0$ in the spaceship’s frame, goes through the origin of our $x$-axis. This happens at the event $O$ which has coordinates $(0, 0)$ in both frames. To measure the length of the spaceship in our frame, we imagine that we have observers at all the points of our $x$-axis who can check whether at $t = 0$ (note: $t = 0$!) the spaceship is passing in front of them. The emphasis on $t = 0$ is to stress that we must measure the length of the spaceship at time equal to zero all along our $x$-axis. We already stated that the beacon at the end of the spaceship is observed at event $O$. Let us denote by $A$ the event where the front of the spaceship is detected by our observer. In our frame $A$ will have coordinates $ct = 0, x = d$, where $d$ is the observed length of the spaceship, which we need to determine. In the frame of the spaceship event $A$ will have coordinates $ct', x' = \ell$, where $ct'$ can be calculated but is irrelevant for us, and $x'$ will be equal to $\ell$ because in the rest frame of the spaceship the length of the ship is $\ell$. Substituting
\(x' = \ell, x = d, t = 0\) into Eq. 31: \(x' = \gamma(x - vt)\), we find

\[\ell = \gamma d\]  \hspace{1cm} (60)

or \(d = \ell/\gamma\). This means that our observers will find that, as the spaceship passes in front of them at \(t = 0\), its measured length \(d\) is shorter by a factor \(1/\gamma = \sqrt{1 - v^2/c^2}\) than its rest length. With a consideration similar to the one we made for the time dilation, the result we just obtained assumes that the length of the spaceship is measured instantaneously at \(t = 0\) (our time.) If, for example, we take instead a picture of the spaceship at \(t = 0\) as its front passes through the origin of our \(x\)-axis, the light from the back of the spaceship will have to be emitted earlier, at negative \(t\), to reach our camera at \(t = 0\), and so the apparent length of the spaceship in the picture will not be given by \(d\). In problems of special relativity it is very important to have a clear notion of all the events involved in the problem, and then use the Lorentz transformation equation to relate their coordinates in different frames.

The squared space-time distance.

Let us consider to events \(A\) and \(B\). We will orient the \(x\)-axis along the spatial separation of the two events. Thus we will have \(y_A = y_B, z_A = z_B\) and the squared space-time distance will be given by

\[d_{AB} = (ct_B - ct_A)^2 - (x_B - x_A)^2\]  \hspace{1cm} (61)

We distinguish three cases according to the sign of \(d_{AB}\).

- \(d_{AB} > 0\) In this case the separation between the two events is called “time-like”. The sign of \(t_B - t_A\) will be the same in all frames of reference. Let us verify this for a boost along the \(x\)-axis (but the argument can be generalized to any boost.) We will have (see Eq. 30)

\[t'_B - t'_A = \gamma \left[(t_B - t_A) - \frac{v}{c^2} (x_B - x_A)\right]\]  \hspace{1cm} (62)

\(v\) being the velocity of the boost. Now, from \(d_{AB} > 0\) it follows that

\[|x_B - x_A| < c|t_B - t_A|\]  \hspace{1cm} (63)

and thus also

\[\left|\frac{v}{c^2} (x_B - x_A)\right| < |t_B - t_A|\]  \hspace{1cm} (64)
since \( v/c \) has always magnitude less than one. As a consequence the absolute value of the last term in the r.h.s. of Eq. 62 will always be smaller than the absolute value of the first term and the sign of the r.h.s. cannot change. Let us then imagine that \( t_B \) is greater than \( t_A \). This means that event \( B \) is in the future with respect to event \( A \). The two events may be causally connected, in the sense that something happening at \( A \) can influence what happens at \( B \). In particular, it is possible to find a reference frame where \( A \) and \( B \) happen in the same point in space. Indeed Eq. 31 gives

\[
x'_B - x'_A = \gamma [(x_B - x_A) - v(t_B - t_A)]
\] (65)

If we choose

\[
v = \frac{x_B - x_A}{t_B - t_A}
\] (66)

which is permissible since on account of Eq. 63 the magnitude of \( v \) will be smaller than the speed of light, then in the boosted frame we will have \( x'_B = x'_A \).

- \( d_{AB} = 0 \) In this case the separation between the two events is called "light-like". A light signal emitted at the event happening earlier in time can be detected at the other event. Light always propagates between events with light-like separation.

- \( d_{AB} < 0 \) In this case the separation between the two events is called "space-like". In this case the spatial distance between the two events will be larger than \( c \) times their time separation in all reference frames. It will not be possible for light or any other signal emitted at event \( A \) to reach event \( B \) or vice versa, and thus the two events cannot be causally connected. Heisenberg would say that \( B \) happens in the present of \( A \) and vice versa. Indeed when the separation between two events is space-like it is possible to find a frame of reference where they happen at the same time. This can also be easily shown. Let us demand that \( t'_A = t'_B \) in Eq. 62 above. We get

\[
\gamma \left[ (t_B - t_A) - \frac{v}{c^2} (x_B - x_A) \right] = 0
\] (67)

which can be solved for \( v \), obtaining

\[
v = c \frac{ct_B - ct_A}{x_B - x_A}
\] (68)
\( v \) will be smaller in magnitude than \( c \) because now
\[
|x_B - x_A| > c|t_B - t_A|
\]  
(69)

Since the magnitude of \( v \) will not exceed \( c \), it will be possible, at least in principle, to boost the reference frame to velocity \( v \), and then, in the boosted frame, \( A \) and \( B \) will occur simultaneously.

**Four-vectors.**

In three-dimensional space given a point \( P \) with coordinates \((x, y, z)\) we associate to it a coordinate vector \( \vec{r} \) with components
\[
\begin{pmatrix} r_1 \\ r_2 \\ r_3 \end{pmatrix} = \begin{pmatrix} x \\ y \\ z \end{pmatrix}
\]  
(70)

The notion of coordinate vector generalizes then to other physical quantities which are characterized by magnitude and direction, as the velocity \( \vec{v} \), the momentum \( \vec{p} \) etc.

In a similar manner, in four-dimensional space-time given an event with space-time coordinates \((ct, x, y, z)\) we associate to it a coordinate four-vector \( \vec{R} \) with components
\[
\begin{pmatrix} R_0 \\ R_1 \\ R_2 \\ R_3 \end{pmatrix} = \begin{pmatrix} ct \\ x \\ y \\ z \end{pmatrix}
\]  
(71)

Note: in order to avoid confusion we use boldface symbols to denote four-vectors, reserving arrow notation, as in \( \vec{r} \), for vectors in three-dimensional space. Also we will use Greek letters to denote the components of four vectors\(^2\).

The notion of coordinate four-vector also generalizes to other quantities, which will play an important role in what follows. One such quantity is the displacement \( \Delta^{(AB)} \) between two events \( A \) and \( B \). Let us denote by

\(^2\)In books and articles on relativity, especially when dealing with general relativity, one will often encounter upper and lower indices, as in \( R_\mu \) and \( R^\mu \) to denote related, but different valued components of a given quantity. We will not need to make such distinction and we will only use lower indices.
\( \mathbf{R}^{(A)} = (ct_A, x_A, y_A, z_A) \) the coordinate four-vector of event \( A \) and by \( \mathbf{R}^{(B)} = (ct_B, x_B, y_B, z_B) \) the coordinate four-vector of event \( B \). Then \( \Delta \) is defined as

\[
\Delta^{(AB)} = \mathbf{R}^{(B)} - \mathbf{R}^{(A)}
\]  

(72)

or explicitly in term of components

\[
\begin{align*}
\Delta_0^{(AB)} &= R_0^{(B)} - R_0^{(A)} = ct_B - ct_A \\
\Delta_1^{(AB)} &= R_1^{(B)} - R_1^{(A)} = x_B - x_A \\
\Delta_2^{(AB)} &= R_2^{(B)} - R_2^{(A)} = y_B - y_A \\
\Delta_3^{(AB)} &= R_3^{(B)} - R_3^{(A)} = z_B - z_A
\end{align*}
\]  

(73)

The vector \( \Delta^{(AB)} \) gives the space-time displacement between two events and we recognize in its components the quantities which entered in the definition of the squared space-time distance in Eq. 4.

Given two four-vectors \( \mathbf{U} \) and \( \mathbf{V} \) their dot-product is defined as

\[
\mathbf{U} \cdot \mathbf{V} = U_0 V_0 - U_1 V_1 - U_2 V_2 - U_3 V_3
\]  

(74)

We see then that the squared space-time distance we defined in Eq. 4 may also be expressed as

\[
d_{AB} = \Delta^{(AB)} \cdot \Delta^{(AB)}
\]  

(75)

In a change of reference frame four-vectors change as the space-time coordinates. So, if for example \( V_\mu \) and \( V'_\mu \) are the components of the four-vector \( \mathbf{V} \) in a frame \( \mathcal{R} \) and in a frame \( \mathcal{R}' \) which moves with rapidity \( \eta \) in frame \( \mathcal{R} \) along the \( x \)-axis, \( V_\mu \) and \( V'_\mu \) will be related by

\[
\begin{pmatrix}
V_0'' \\
V_1'' \\
V_2'' \\
V_3''
\end{pmatrix} =
\begin{pmatrix}
\cosh \eta & -\sinh \eta & 0 & 0 \\
-\sinh \eta & \cosh \eta & 0 & 0 \\
0 & 0 & 1 & 0 \\
0 & 0 & 0 & 1
\end{pmatrix}
\begin{pmatrix}
V_0' \\
V_1' \\
V_2' \\
V_3'
\end{pmatrix}
\]  

(76)

or, equivalently, in terms of the velocity \( v = c \tanh \eta \), and \( \gamma = 1/\sqrt{1 - v^2/c^2} \)

\[
\begin{pmatrix}
V_0'' \\
V_1'' \\
V_2'' \\
V_3''
\end{pmatrix} =
\begin{pmatrix}
\gamma & -\gamma v/c & 0 & 0 \\
-\gamma v/c & \gamma & 0 & 0 \\
0 & 0 & 1 & 0 \\
0 & 0 & 0 & 1
\end{pmatrix}
\begin{pmatrix}
V_0' \\
V_1' \\
V_2' \\
V_3'
\end{pmatrix}
\]  

(77)
Of course the boost does not have to be along the \( x \)-axis. We would have similar formulae, with the obvious exchanges of rows and columns, for boosts along the \( y \)-axis or the \( z \)-axis. And, also of course, we would have coordinate transformations for rotations around any of the three space axes. These do not affect the time variable and are identical to the transformations induced by rotations in three-dimensional space. For example, if \( R' \) is obtained rotating \( R \) by an angle \( \phi \) around the \( z \)-axis, then the four-vector components will be related by

\[
\begin{pmatrix}
V'_0 \\
V'_1 \\
V'_2 \\
V'_3
\end{pmatrix}
= \begin{pmatrix}
1 & 0 & 0 & 0 \\
0 & \cos \phi & \sin \phi & 0 \\
0 & -\sin \phi & \cos \phi & 0 \\
0 & 0 & 0 & 1
\end{pmatrix}
\begin{pmatrix}
V_0 \\
V_1 \\
V_2 \\
V_3
\end{pmatrix}
\]  

(78)

If one performs several changes of frame of reference the corresponding matrices are multiplied. The outcome will be of the form

\[ V' = M V \]  

(79)

where \( M \) is the result of all the matrix multiplications. The very important point is that, since all individual transformations preserve the squared-length of the vectors, or equivalently the dot product defined in Eq. 74, so will \( M \).

For any two four-vectors \( U, V \) we will have

\[ U' \cdot V' = (MU) \cdot (MV) = U \cdot V \]  

(80)

The set \( G \) of all \( 4 \times 4 \) matrices \( M \) that preserve the dot-product defined above, i.e. the set of all matrices \( M \) for which Eq. 79 is satisfied for all four-vectors \( U, V \), form what mathematicians call a “group”. A group is a set of elements (in our case the \( 4 \times 4 \) matrices \( M \)) with a product operation (in our case the matrix product) with the properties that the product of any two elements is still in the set (in our case if \( M \) and \( M' \) preserve the dot product, so does their product \( MM' \)), that there is an identity (in our case, of course, the identity matrix), and that the inverse of any element is also in the set (in our case if \( M \) preserves the dot-product so does \( M^{-1} \).) \( G \) is called the Lorentz group. It is a group with six independent transformations, which we may take to be the three rotations around the space axes and the three boosts along the space axes. All other frame of reference changes can be obtained as a product of the above.
The velocity four-vector.

Let us consider a situation where a point-like moves in space-time. At a
certain moment of time $t$ it is detected at position $\vec{r}$. This is event $A$. The
space-time coordinates of $A$ are $(ct, x, y, z)$. We are interested in measuring
the velocity of the object, so we detect its position again a very short time
later, indeed a time so short that we can consider it infinitesimal and denote
it by $dt$. The new detection is event $B$ and the space-time coordinates of $B$
are $(c(t + dt), x + dx, y + dy, z + dz)$. There is nothing abstruse in all of this.
It is really simple: what we are saying is that in our frame of reference at
time $t$ the object is at position $x, y, z$ and a moment later, at time $t + dt$ the
object is at position $x + dx, y + dy, z + dz$. In these circumstances we would
measure the components of the object’s velocity to be

$$
v_1 = \frac{dx}{dt}
v_2 = \frac{dy}{dt}
v_3 = \frac{dz}{dt}
$$

or, with vector notation

$$
\vec{v} = \frac{d\vec{r}}{dt}
$$

The problem with all of this is that an observer in a boosted frame $\mathcal{R}'$
would measure a velocity $\vec{v}'$ that would have a rather complicated relation
with $\vec{v}$. Ideally we would like the velocity to be a four-vector, like a
displacement, and transform like a four-vector, i.e. according to Eqs. 76 or 77, in
the change of reference frame from $\mathcal{R}$ to the boosted frame $\mathcal{R}'$. In order
to achieve this, let us backtrack a little and focus on the space-time displacement
between the events $A$ and $B$. Let us denote by $\mathbf{R}^{(A)}$ the coordinate four-vector of event
$A$ and by $\mathbf{R}^{(B)}$ the coordinate four-vector of event $B$. Since the two events
are infinitesimally close, the displacement four-vector between the two will
be infinitesimal and we will denote in by $d\mathbf{R}$:

$$
d\mathbf{R} = \mathbf{R}^{(B)} - \mathbf{R}^{(A)}
$$
The components of $d\mathbf{R}$ are

\begin{align*}
    dR_0 &= c \, dt \\
    dR_1 &= dx \\
    dR_2 &= dy \\
    dR_3 &= dz
\end{align*}

and its squared-length is

$$d\mathbf{R} \cdot d\mathbf{R} = c^2 \, dt^2 - dx^2 - dy^2 - dz^2$$

(85)

Note that $d\mathbf{R}$ is time-like because the two events $A$ and $B$ connected by $d\mathbf{R}$ represent two subsequent (infinitesimally close) positions of the same object in motion. Also, the two events are obviously causally connected. So, the squared length $d\mathbf{R} \cdot d\mathbf{R}$ is positive. Taking its square root and dividing by $c$ we obtain a quantity with dimension of time, which we denote by $d\tau$ and will call “the infinitesimal proper time interval”

$$d\tau = \sqrt{dt^2 - \frac{dx^2 + dy^2 + dz^2}{c^2}} = \sqrt{dt^2 - \frac{(d\vec{r})^2}{c^2}}$$

(86)

To understand the significance of $d\tau$ and why we call it infinitesimal proper time interval, let us imagine that the object we are considering moves with constant velocity $\vec{v}$. In this case we do not need $A$ and $B$ to be infinitesimally close, but we may imagine them to be subsequent observations of the object in motion at a finite time interval $\Delta t$ (time measured in our reference frame.) The displacement four vector, which we now denote by $\Delta \mathbf{R}$, will be finite and will be given by

$$\Delta \mathbf{R} = \mathbf{R}^{(B)} - \mathbf{R}^{(A)}$$

(87)

Its squared-length will be

$$\Delta \mathbf{R} \cdot \Delta \mathbf{R} = c^2 (\Delta t)^2 - (\Delta x)^2 - (\Delta y)^2 - (\Delta z)^2 = c^2 (\Delta t)^2 - (\Delta \vec{r})^2$$

(88)

where we denoted by $\Delta \vec{r}$ the spatial displacement between $A$ and $B$ in our frame. Taking the square root and dividing by $c$ we obtain now a finite proper time interval

$$\Delta \tau = \sqrt{(\Delta t)^2 - \frac{(\Delta \vec{r})^2}{c^2}}$$

(89)
Now, $\Delta \tau$ is an invariant quantity and it is the same in all reference frame (because the dot product of four-vectors is invariant.) If we go to the reference frame $\mathcal{R}'$ of the object in motion, i.e. its rest frame, the spatial coordinates of $A$ and $B$ will be the same and thus $\Delta \vec{r}'$ will be zero. This means that in the frame of the moving object

$$\Delta \tau = \Delta t'$$

(90)

i.e. the proper time interval reduces to the actual time interval in the frame of the moving object. It is interesting to relate the expressions for $\Delta \tau$ in our frame $\mathcal{R}$ and the frame $\mathcal{R}'$ of the moving object. In our frame the object moves with velocity $\vec{v}$ and thus

$$d\vec{r} = \vec{v} \Delta t$$

(91)

Inserting into Eq. 89 we get

$$\Delta \tau = \sqrt{(\Delta t)^2 - \frac{(\vec{v})^2(\Delta t)^2}{c^2}} = \sqrt{1 - \frac{v^2}{c^2}} \Delta t = \frac{\Delta t}{\gamma}$$

(92)

i.e. the proper time interval $\Delta \tau$ between $A$ and $B$ is shorter by a factor $1/\gamma$ with respect to the interval $\Delta t$ measured in our frame. But this corresponds precisely to what we know from time dilation. The clock of the moving object, which we are going to call a spaceship from now on, ticks slower than our clock.

And how about the situation when the motion of the spaceship is not uniform? Then $d\tau$ still measures the infinitesimal amount of time elapsed according to the spaceship clock, but in order to find the finite amount of elapsed time one must integrate $d\tau$ along the spaceship’s motion. We illustrate this with the example of a spaceship moving with constant acceleration. One can show that the trajectory specified in our frame by the equation

$$x(t) = \frac{c^2}{a} \sqrt{1 + \frac{a^2}{c^2} t^2}$$

(93)

describes the motion of a spaceship which at $t = 0$ leaves from $x = c^2/a$ with zero initial velocity and then moves along the $x$-axis picking up speed with constant acceleration $a$ in its own rest frame. In this case we must first calculate

$$dx = \frac{1}{\sqrt{1 + a^2 t^2 / c^2}} dt$$

(94)
then
\[ d\tau^2 = dt^2 - \frac{dx^2}{c^2} = \left(1 - \frac{a^2t^2}{c^2 + a^2t^2}\right) dt^2 = \frac{c^2}{c^2 + a^2t^2} dt^2 \] (95)
and finally
\[ d\tau = \frac{c}{\sqrt{c^2 + a^2t^2}} dt \] (96)
This equation must now be integrated to find the proper time, i.e. the time aboard the spaceship, as a function of \( t \). Assuming \( \tau = 0 \) for \( t = 0 \) we get
\[ \int_0^\tau d\tau' = \tau = \int_0^t \frac{c}{\sqrt{c^2 + a^2t'^2}} dt' \] (97)
The integral can done with a change of variable. Let us set
\[ t = \frac{c \sinh \alpha}{a} \] (98)
With this we get
\[ dt = \frac{c \cosh \alpha}{a} d\alpha \] (99)
and
\[ \sqrt{c^2 + a^2t^2} = \sqrt{c^2(1 + \sinh^2 \alpha)} = c \cosh \alpha \] (100)
so that the whole integral in Eq. 97 reduces to
\[ \tau = \int_0^\alpha \frac{c}{a} d\alpha' = \frac{c}{a} \alpha \] (101)
or, going back to the variable \( t \)
\[ \tau = \frac{c}{a} \text{arcsinh} \frac{at}{c} \] (102)
The inverse sinh function, arcsinh, can be expressed in terms of natural logs\(^3\), but perhaps more interesting is to solve Eq. 102 for \( t \) which gives
\[ t = \frac{c}{a} \sinh \frac{a \tau}{c} \] (103)
\(^3\)From \( \sinh \alpha = (e^\alpha - e^{-\alpha})/2 = x \), setting \( y = e^\alpha \) we obtain the equation for \( y \):
\[ (y - y^{-1})/2 = x \] or \( y^2 - 2yx - 1 = 0 \) with solution \( y = x + \sqrt{x^2 + 1} \). With \( \alpha = \log y \) we thus get \( \alpha = \text{arcsinh} x = \log(x + \sqrt{x^2 + 1}) \).
expressing our time as function of the time aboard the spaceship.

We are now in the position of defining the velocity four-vector. The idea is that, instead of dividing the infinitesimal space-time displacement $d\mathbf{R}$ (see Eqs. 83-85) by $dt$ we will divide it by the proper time increment $d\tau$. We thus define the velocity four-vector $\mathbf{U}$ as

$$\mathbf{U} = \frac{d\mathbf{R}}{d\tau} \quad (104)$$

or, in terms of components,

$$U_0 = \frac{cdt}{d\tau}, \quad U_1 = \frac{dx}{d\tau}, \quad U_2 = \frac{dy}{d\tau}, \quad U_3 = \frac{dz}{d\tau} \quad (105)$$

We should note, and remember, that the magnitude of the velocity four-vector is $c$. Indeed we have

$$\mathbf{U} \cdot \mathbf{U} = \frac{c^2 dt^2 - dx^2 - dy^2 - dz^2}{d\tau^2} = c^2 \quad (106)$$

since $d\tau^2 = dt^2 - (dx^2 + dy^2 + dz^2)/c^2$ (see Eq. 86).

The proper time increment $d\tau$ is invariant in a change of reference frame. It is the same in all frames. Therefore the transformation of the components $U_\mu$ in a change of frame will be the same as the transformation of the components of $d\mathbf{R}$, i.e. the same as those of all four-vectors. If we denote by $U'_\mu$ the components of the velocity four-vector in the frame $\mathcal{R}'$ which moves along the $x$-direction of frame $\mathcal{R}$ with rapidity $\eta$, or velocity $v = c \tanh \eta$, we will have

$$\begin{pmatrix} U'_0 \\ U'_1 \\ U'_2 \\ U'_3 \end{pmatrix} = \begin{pmatrix} \cosh \eta & -\sinh \eta & 0 & 0 \\ -\sinh \eta & \cosh \eta & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} U_0 \\ U_1 \\ U_2 \\ U_3 \end{pmatrix} \quad (107)$$
or, equivalently,

\[
\begin{pmatrix}
U'_0 \\
U'_1 \\
U'_2 \\
U'_3
\end{pmatrix} =
\begin{pmatrix}
\gamma & -\gamma v/c & 0 & 0 \\
-\gamma v/c & \gamma & 0 & 0 \\
0 & 0 & 1 & 0 \\
0 & 0 & 0 & 1
\end{pmatrix}
\begin{pmatrix}
U_0 \\
U_1 \\
U_2 \\
U_3
\end{pmatrix}
\]  

(108)

where \( \gamma = \gamma(v) = 1/\sqrt{1 - v^2/c^2} \).

The velocity four-vector can be expressed in terms of the ordinary, three dimensional velocity vector. We will denote the latter by \( \vec{u} \), reserving the symbol \( v \) for the velocity of frame \( \mathcal{R}' \) with respect to frame \( \mathcal{R} \). We have indeed

\[
dx = u_1 \, dt
\]
\[
dy = u_2 \, dt
\]
\[
dz = u_3 \, dt
\]

(109)

and

\[
d\tau = \sqrt{dt^2 - (dx^2 + dy^2 + dz^2)/c^2}
\]
\[
= \sqrt{dt^2 - (u_1^2 + u_2^2 + u_3^2) \, dt^2/c^2}
\]
\[
= \sqrt{1 - \vec{u}^2/c^2} \, dt = \frac{dt}{\gamma(u)}
\]

(110)

where we used

\[
\gamma(u) = \frac{1}{\sqrt{1 - u^2/c^2}}
\]

(111)

Substituting Eqs. 109, 110 into Eq. 105 we find

\[
U_0 = \gamma(u)c
\]
\[
U_1 = \gamma(u)u_1
\]
\[
U_2 = \gamma(u)u_2
\]
\[
U_3 = \gamma(u)u_3
\]

(112)

These equations are of course consistent with the fact that the magnitude of \( \mathbf{U} \) is \( c \) since

\[
\gamma(u)^2(c^2 - u_x^2 - u_y^2 - u_z^2) = \frac{1}{1 - u^2/c^2} (c^2 - u^2) = c^2
\]

(113)
Using the expressions for the four-velocity components \( U_\mu \) and \( U'_\mu \) in terms of the three-dimensional velocity components \( u_i \) and \( u'_i \) in the reference frames \( \mathcal{R} \) and \( \mathcal{R}' \), and the transformation equations for the four-velocity components given by Eq. 108, one can find the relation between the velocities \( u_x, u_y, u_z \) and \( u'_x, u'_y, u'_z \) in the two frames. This is left as an exercise for the reader. 

(Hint: Substitute \( U_0, U_1, U_2, U_3 \) from Eq. 112 into Eq. 108, divide lines 2, 3, and 4 in the resulting equation by line 1 and simplify.)

**The energy-momentum four-vector.**

In non-relativistic physics the momentum of an object of mass \( m \) moving with velocity \( \vec{u} \) is defined as

\[
\vec{p} = m \vec{u}
\]  

(114)

The total momentum of an isolated system is conserved. If \( m_i, \vec{u}_i, \vec{p}_i \) and \( \tilde{m}_j, \tilde{\vec{u}}_j, \tilde{\vec{p}}_j \) are the masses, velocities and momenta of its components before and after some reaction, then

\[
\sum_i \vec{p}_i = \sum_j \tilde{\vec{p}}_j
\]  

(115)

Notice that we do not assume that the masses \( m_i, \tilde{m}_j \) are identical and not even that the number of the components before and after the reaction are the same: for example one of the objects could fragment into two or more. However the total mass will be conserved

\[
\sum_j \tilde{m}_j = \sum_i m_i
\]  

(116)

Momentum conservation is invariant with respect to a non-relativistic change of reference frame. In a frame \( \mathcal{R}' \) which moves with velocity \( \vec{v} \) with respect to the original frame \( \mathcal{R} \) the velocities of the components before and after the reaction are given by

\[
\vec{u}'_i = \vec{u}_i - \vec{v}
\]

\[
\tilde{\vec{u}}'_j = \tilde{\vec{u}}_j - \vec{v}
\]  

(117)

and, correspondingly,

\[
\vec{p}'_i = \vec{p}_i - m_i \vec{v}
\]

\[
\tilde{\vec{p}}'_j = \tilde{\vec{p}}_j - \tilde{m}_j \vec{v}
\]  

(118)
As a consequence
\[ \sum_j \tilde{p}_j = \sum_j \tilde{p}_j - (\sum_j \tilde{m}_j)\tilde{v} = \sum_i \tilde{p}_i - (\sum_i m_i)\tilde{v} = \sum_i p'_i \]
\[ (119) \]
where we used Eqs. 115 and 116 to go from the second to the third term in the above equality. Note that beyond momentum conservation also the conservation of the total mass of the system is crucial for demonstrating that momentum conservation is invariant in a change of reference frame.

In relativistic dynamics, however, if we define the momentum of a moving object by Eq. 114, i.e. as \( \tilde{p} = m\tilde{u} \), because of the non-linear way in which three-dimensional velocity transforms in a change of reference frame, if momentum is conserved in frame \( \mathcal{R} \), generally it will not be conserved in a boosted frame \( \mathcal{R}' \). In order to overcome the difficulty, we define the four-vector momentum \( \mathbf{P} \) of an object of mass \( m \) by
\[ \mathbf{P} = m\mathbf{U} \]
\[ (120) \]
where \( \mathbf{U} \) is the velocity four-vector of the object. It is easy to prove that if the total momentum four-vector is conserved in some reaction in a reference frame \( \mathcal{R} \), it will be conserved in every other reference frame \( \mathcal{R}' \). Indeed, the velocity components of \( \mathbf{U} \) and \( \mathbf{U}' \) are related by
\[ \mathbf{U}' = M\mathbf{U} \]
\[ (121) \]
where we used compact vector, matrix notation, and \( M \) (not to be confused with the mass \( m \)) is the \( 4 \times 4 \) Lorentz transformation matrix which implements the change of four-vector components in going from \( \mathcal{R} \) to \( \mathcal{R}' \). It follows that the momentum components also transform by
\[ \mathbf{P}' = M\mathbf{P} \]
\[ (122) \]
As a consequence if the total four-momentum is conserved in the reference frame \( \mathcal{R} \), i.e. if
\[ \sum_j \tilde{P}_j = \sum_i P_i \]
\[ (123) \]
where \( P_i \) and \( \tilde{P}_j \) are the four-momenta of the system components before and after the reaction, then the total four-momentum will also be conserved in the reference frame \( \mathcal{R}' \), i.e.
\[ \sum \tilde{P}'_j = \sum P'_i \]
\[ (124) \]
Indeed from Eq. 122 it follows that

$$\sum_j \vec{P}_j' = M \sum_j \vec{P}_j$$

(125)

and

$$\sum_i P_i' = M \sum_i P_i$$

(126)

Thus we will have

$$\sum_j \vec{P}_j' = M \sum_j \vec{P}_j = M \sum_i P_i = \sum_i P_i'$$

(127)

where we used the momentum conservation in frame $\mathcal{R}$ (Eq. 123).

Granted that conservation of four-momentum, as defined above, is invariant with respect of a change of reference frame, it remains to be established what is the meaning of this four-vector. Let us begin by relating its components to the three-dimensional velocity $\vec{u}$ and momentum $\vec{p}$ of an object of mass $m$. $\vec{U}$ and $\vec{u}$ are related by (see Eq. 112)

\begin{align*}
U_0 &= \gamma(u) c \\
U_1 &= \gamma(u) u_1 \\
U_2 &= \gamma(u) u_2 \\
U_3 &= \gamma(u) u_3
\end{align*}

(128) (129) (130) (131)

From these equations and Eq. 120 we obtain

\begin{align*}
P_0 &= \gamma(u) mc = \frac{mc}{\sqrt{1 - u^2/c^2}} \\
P_1 &= \gamma(u) mu_1 = \frac{p_1}{\sqrt{1 - u^2/c^2}} \\
P_2 &= \gamma(u) mu_2 = \frac{p_2}{\sqrt{1 - u^2/c^2}} \\
P_3 &= \gamma(u) mu_3 = \frac{p_3}{\sqrt{1 - u^2/c^2}}
\end{align*}

(132) (133) (134) (135)

So, in the non-relativistic limit where $u \ll c$ and $\gamma(u) \approx 1$ $P_0$ reduces to the mass of the object multiplied by the speed of light, while the spatial components of $\vec{P}$ reduce to the non-relativistic momentum components $p_i$. 25
The conservation of the total four-momentum in an isolated system is then, at least, consistent with the conservation of mass and momentum in non-relativistic dynamics. Experiment, especially in the context of sub-atomic particle reactions, shows that the conservation of the total four-momentum of an isolated system remains true also in relativistic dynamics. Nevertheless we still must develop an understanding of the physical significance of the first component \( P_0 \). One clue comes from expanding \( P_0 \) to second order in \( u/c \).

We have

\[
P_0 = mc(1 - u^2/c^2)^{-1/2} = mc + \frac{mu^2}{2c} + \ldots
\]  

or, equivalently,

\[
P_0c = mc^2 + \frac{mu^2}{2} + \ldots
\]

where we recognize in the second term at the r.h.s. the non-relativistic kinetic energy of an object of mass \( m \) in motion with velocity of magnitude \( u \). This leads us to associate \( P_0c \) with the energy of the object in motion, at least insofar as the excess over \( mc^2 \) is concerned. But the separation of \( P_0c \) into a mass term and an energy term is not tenable, especially since experiment shows that while in an isolated system the total \( P_0c \) is conserved, the total mass of the components is not necessarily conserved. For example, the \( \pi_0 \) meson undergoes the (rare) decay into an \( e^-e^+ \) (electron, positron) pair, where the \( \pi_0 \) mass of approx. 135 MeV/c\(^2\) is much larger than the combined mass of approx. 1.22 MeV/c\(^2\) of electron and positron. So we are led to the inescapable conclusion that all of \( P_0c \), including the mass term \( mc^2 \), must be identified with the energy \( E \) of the object:

\[
E = P_0c = \gamma(u)mc^2
\]

In particular, in the rest frame of the object, where \( \gamma(u) = 1 \),

\[
E = mc^2
\]

This is the famous \( E = mc^2 \) relation between mass and energy, amply verified by phenomena, for example at the nuclear level, where, in the decay of a heavy nucleus, the initial mass of the nucleus exceeds the total mass of the products of the decay, the excess mass being converted in the kinetic energy of the decay products. For a moving object with non-vanishing spatial four-momentum components \( P_i \), the relation between energy, rest mass and
momentum can be inferred from $P = mU$ and the fact that the four-velocity $U$ has squared length equal to $c^2$. This implies

$$P \cdot P = m^2 U \cdot U = m^2 c^2$$

or

$$P_0^2 - \vec{P}^2 = m^2 c^2$$

where we used the notation $\vec{P}^2$ for $P_1^2 + P_2^2 + P_3^2$. In terms of energy and momentum, the relation reads

$$E^2 - \vec{P}^2 c^2 = m^2 c^4$$

With the identification of the energy $E$ of an object with $c$ times the zero component of its momentum, conservation of four-momentum in an isolated system becomes conservation of energy and momentum: in an isolated system both relativistic energy and relativistic momentum are conserved. It is worth noting, at this point, that in a change of reference frame $P_0$ and the $P_i$ components of four-momentum are mixed. For example, in a boost of velocity $v$ along the $x$-axis we will have

$$P_0' = \gamma(v) P_0 - \gamma(v) \frac{v}{c} P_1$$

$$P_1' = -\gamma(v) \frac{v}{c} P_0 + \gamma(v) P_1$$

We see then that conservation of total energy (total $P_0 c$) as well as of total momentum, in frame $\mathcal{R}$, are crucial to insure conservation of total energy and momentum in frame $\mathcal{R}'$.

The equivalence between mass and energy works also the other way around. Energy carries an intrinsic mass. To appreciate this let us consider the following situation: two objects of identical mass move in opposite directions along the $x$-axis of a frame of reference $\mathcal{R}$ with opposite velocities $u$ and $-u$. The total four-momentum has components

$$P_0 = 2\gamma(u)mc = \frac{mc}{\sqrt{1 - u^2/c^2}}$$

$$P_1 = \gamma(u)mu + \gamma(u)m(-u) = 0$$

$$P_2 = 0$$

$$P_3 = 0$$
If the two objects move apart with very large velocity, $P_0$ will be much larger than $2mc$. If we boost now the whole system along the $x$-axis to velocity $v$ (which is equivalent to changing the frame $\mathcal{R}$ to a frame moving with velocity $-v$ along the $x$-axis,) the system will acquire a non-vanishing total momentum

$$P'_1 = \gamma(v) \frac{v}{c} P_0 = \gamma(v) v (2m \gamma(u))$$ (149)

If we are to interpret this value of $P'_1$ as the momentum of an object of mass $M$ moving with velocity $v$, we see that $M$ must be equal to $2m \gamma(u)$, i.e. much larger than the combined mass of the moving objects. The conclusion is that the energy $E = P_0 c$ carries a mass $M = E/c^2$, which will manifest itself in the non-vanishing total momentum of the system when it is boosted to velocity $v$. Of course, the two objects flying apart could be the result of the decay of some heavier object at rest in the original frame. Then the mass of this object would be precisely $M$ and, in the decay, the corresponding energy $Mc^2$ would be converted into the mass and kinetic energy of the products of the decay.

As a final observation we go back to the fact that a photon of frequency $\nu$ carries an energy $E = h\nu$. The theory of electromagnetism tells us that electromagnetic radiation propagating in a certain direction carries a momentum oriented in the same direction and with magnitude $P = E/c$. We conclude that the photon will have momentum of magnitude $h\nu/c$. Thus, for the spatial components of the photon’s four-momentum we will have

$$|\vec{P}| = \sqrt{P_1^2 + P_2^2 + P_3^2} = \frac{E}{c} = \frac{h\nu}{c}$$ (150)

The relation $m^2 c^4 = E^2 - \vec{P}^2 c^2$ gives us then

$$m^2 = \frac{(h\nu)^2}{c^4} - \frac{(h\nu/c)^2}{c^2} = 0$$ (151)

The photon has zero mass. Indeed only a zero mass particle can propagate at the speed of light.