Solve all three problems. Give a clear explanation of your work. Correct solutions will be credited with 35 points each for problems 1 and 3, and 30 points for problems 2. Points will be taken away for errors according to the severity of the error.

The solutions to the problems must be written clearly and, if they involve calculations, they must contain a few lines of explanation of the work. Points will be taken away for solutions which just show equations with no interconnecting words indicating the reasoning followed in solving the problem.

The exam is a closed book exam: books and notes must be stored away, as well as all devices that can access the internet, which must be turned off.

Useful data and equations:
(The numerical values have been rounded off to simplify the calculations.)

The kinetic energy of a particle like a proton or an atom at temperature $T$ is given by

$$ E_k = \frac{3}{2} kT $$

The value of $kT$ at 300 K (degree Kelvin) is 0.025 eV

The radius $r_s$ of the sun is $7 \times 10^8$ m

The mass $M_s$ of the sun is $2 \times 10^{30}$ kg

The mass $m_p$ of a proton is $10^9$ eV/c$^2$

The speed $c$ of light $3 \times 10^8$ m/s

The total power per unit area radiated by an object of temperature $T$ is given by

$$ R = \sigma'(kT)^4 $$

(Stefan-Boltzmann law), where

$$ \sigma' = 10^9 \text{W} \cdot \text{m}^{-2} \cdot (\text{eV})^{-4} $$
The Lorentz transformation equations for the change of space-time coordinates from a frame $\mathcal{R}$ to a frame $\mathcal{R}'$ moving with velocity $v$ along the $x$-axis of $\mathcal{R}$ are

\begin{align*}
 ct' &= \gamma ct - \beta \gamma x \quad (4) \\
 x' &= -\beta \gamma ct + \gamma x \quad (5) \\
 y' &= y \quad (6) \\
 z' &= z \quad (7)
\end{align*}

where

\begin{align*}
 \beta &= \frac{v}{c} \\ 
 \gamma &= \frac{1}{\sqrt{1 - \beta^2}} 
\end{align*}

(8) (9)

The components of the momentum four-vector $P_\mu$ of a particle of rest mass $m$ traveling with velocity $\vec{u}$ are given by

\begin{align*}
 P_0 &= \gamma(u)mc \quad (11) \\
 P_1 &= \gamma(u)mu_1 \quad (12) \\
 P_2 &= \gamma(u)mu_2 \quad (13) \\
 P_3 &= \gamma(u)mu_3 \quad (14)
\end{align*}

where

\[ \gamma(u) = \frac{1}{\sqrt{1 - \vec{u}^2/c^2}} \]

(15)

The zero component of the four-momentum vector is related to energy $E$ of the particle by $E = P_0c$.

A photon of energy $E$ has momentum with magnitude $E/c$. In particular, if it travels in the positive $x$ direction, the components of its four-vector momentum will be $P_0 = E/c, P_1 = E/c, P_2 = P_3 = 0$.

In a Lorentz transformation the components of the four-momentum vector transform as the components of the space-time coordinates $ct, x, y, z$.

The mass of a $\pi^0$ meson is $m_{\pi^0} = 135\text{MeV}/c^2$. 

2
Problem 1

Suppose that the sun was formed as a hot ball of hydrogen at some very high temperature. Estimate the time for the sun to cool to 6000 K if there were no integral energy source.

In this problem you may assume that the sun is made of a (very large), number \( N \) of protons in thermal motion, and that the radius \( r_s \) of the sun, as well as \( N \), stay constant during the cooling. The value of temperature when the sun was formed is largely irrelevant, but if you want a definite value you may take it to be \( T_i = 2 \times 10^6 \) K. The actual age of the sun is six order of magnitude greater than the cooling time you will find, which assumes no internal energy source. The sun is heated by proton fusion, which converts matter into thermal energy.

(This is Example 3-11 in the textbook with some added notes.)

Solution

The total power \( P \) radiated by the sun is obtained from the Stefan-Boltzmann law and is given by

\[
P = - \frac{dE_k}{dt} = 4\pi r_s^2 \sigma'(kT)^4
\]  

(16)

\( P \) depends on time because the temperature \( T \) depends on time. The total kinetic energy of the \( N \) particles in the sun is proportional to the temperature

\[
E_k \approx \frac{3}{2} N k T
\]  

(17)

Taking the derivative with respect to time we get

\[
\frac{dE_k}{dt} \approx \frac{3}{2} N k \frac{dT}{dt}
\]  

(18)

From this relation and Eq. 16 we obtain

\[
dt \approx -\frac{\frac{3}{2} N k \frac{dT}{dt}}{4\pi r_s^2 \sigma'(kT)^4}
\]  

(19)

Integrating we find that the time \( \Delta t \) for the sun to cool from some initial
temperature $T_i$ to $T_f = 6000$ K is

$$\Delta t \approx -\frac{3N}{8\pi r^2\sigma'k^3} \int_{T_i}^{T_f} dT \frac{1}{T^4}$$

$$= -\frac{3N}{8\pi r^2\sigma'k^3} \left[ -\frac{1}{T^3} \right]_{T=T_i}^{T=T_f}$$

$$\approx \frac{3N}{8\pi r^2\sigma'(kT_f)^3}$$

where we neglected $1/T_i^3 \ll 1/T_f^3$.

We estimate the number $N$ of protons in the sun to be the mass of the sun divided by the mass of the proton

$$N \approx \frac{M_s}{m_p}$$

which gives

$$\Delta t \approx \frac{M_sc^2}{8\pi r^2 m_pc^2\sigma'(kT_f)^3}$$

$$\approx \frac{(2 \times 10^{30} \text{kg})(3 \times 10^8 \text{m/s})^2}{8\pi(7 \times 10^8 \text{m})^2(10^9 \text{eV})(10^9 \text{W} \cdot \text{m}^{-2} \cdot \text{eV}^{-4})(0.5 \text{eV})^3}$$

$$\approx 10^{11} \text{s}$$

(22)

(where we used $kT_f = k \times 6000 \text{K} = 20 \times k \times 300 \text{K} = 20 \times 0.025 \text{eV} = 0.5 \text{eV}$)

i.e. a mere 3000 years.

**Problem 2**

An observer measures the velocity of two electrons and finds that one has a speed $c/2$ in the $x$ direction and the other has a speed $c/2$ in the $y$ direction. What is the relative speed of the two electrons

(This is Problem 4-11 in the textbook, solved in the discussion sessions.)

**Solution**

We go to the rest frame $\mathcal{R}'$ of the electron moving in the $x$-direction. This frame moves with velocity $v = c/2$ along the $x$-axis of the original frame and
its space-time coordinates, which we denote by a prime, are related to the coordinates in the original frame $\mathcal{R}$ by

\begin{align*}
ct' &= \gamma ct - \beta \gamma x \quad (23) \\
x' &= -\beta \gamma ct + \gamma x \\
y' &= y \quad (25) \\
z' &= z \quad (26)
\end{align*}

where $\beta = v/c = (c/2)/c = 1/2$, $\gamma = 1/\sqrt{1 - \beta^2} = 1\sqrt{3}/4 = 2/\sqrt{3}$. The relative velocity of the two electrons will be given by the velocity of the second electron in the frame $\mathcal{R}'$. We obtain the components of this velocity by taking the differentials of Eqs. 23–25:

\begin{align*}
ct' &= \gamma c dt - \beta \gamma dx \\
dx' &= -\beta \gamma c dt + \gamma dx \quad (28) \\
dy' &= dy \quad (29)
\end{align*}

(The $z$-components are irrelevant.) By dividing Eqs. 28 and 29 by Eq. 27 we get

\begin{align*}
\frac{dx'}{ct'} &= \frac{u_x'}{c} = \frac{-\beta c + dx/dt}{c - \beta dx/dt} = \frac{-\beta c + u_x}{c - \beta u_x} = \frac{-v + u_x}{1 - vu_x/c^2} \quad (30) \\
\frac{dy'}{ct'} &= \frac{u_y'}{c} = \frac{dy/dt}{\gamma(c - \beta dx/dt)} = \frac{u_y}{\gamma(c - \beta u_x)} \quad (31)
\end{align*}

or

\begin{align*}
    u_x' &= -v + u_x \quad (32) \\
    u_y' &= \frac{u_y}{\gamma(1 - vu_x/c^2)} \quad (33)
\end{align*}

where we denoted by $u_x, u_y$ and $u_x', u_y'$ the components of the velocity of the second electron in frames $\mathcal{R}$ and $\mathcal{R}'$ respectively. With $u_x = 0, u_y = c/2$ these equations reduce to

\begin{align*}
    u_x' &= -v = -\frac{c}{2} \quad (34) \\
    u_y' &= \frac{u_y}{\gamma} = \frac{c/2}{2/\sqrt{3}} = \frac{c\sqrt{3}}{4} \quad (35)
\end{align*}
Finally we find for the relative velocity

\[ v_{rel} = \sqrt{u_x'^2 + u_y'^2} = \sqrt{\frac{1}{4} + \frac{3}{16}} c = \frac{\sqrt{7}}{4} c \]  \hspace{1cm} (36)

Alternatively one can solve the problem as follows:

We denote by \( \vec{u} \) and \( \vec{u}' \) the velocities of the second electron in the frames \( \mathcal{R} \) and \( \mathcal{R}' \), and by \( u \) and \( u' \) the magnitude of these velocities. In \( \mathcal{R} \) \( \vec{u} = (0, c/2, 0) \) and thus \( u = c/2 \). In \( \mathcal{R}' \) the value of \( u' \), which is the relative speed of the two electrons, remains to be determined. Let us consider now the four-component velocities \( \mathbf{U} \) and \( \mathbf{U}' \) of this electron in the two frames. We have

\[
\begin{align*}
U_0 &= \gamma(u)c = \frac{1}{\sqrt{1 - (c/2)^2/c^2}} c = \frac{1}{\sqrt{3/4}} c \\
U_1 &= 0 \\
U_0' &= \gamma(u)c = \frac{1}{\sqrt{1 - u'^2/c^2}} c \\
\end{align*}
\]  \hspace{1cm} (37, 38, 39)

The other components of the four velocity are irrelevant for this problem. Now, from the Lorentz transformation equations we get

\[
U_0' = \gamma(v)U_0 - \frac{v\gamma(v)}{c} U_1 = \gamma(v)U_0
\]  \hspace{1cm} (40)

where \( v \) is the velocity of frame \( \mathcal{R}' \) in frame \( \mathcal{R} \). Since \( v \) is also equal to \( c/2 \), \( \gamma(v) \) is equal to \( \gamma(u) \) namely

\[
\gamma(v) = \frac{1}{\sqrt{3/4}}
\]  \hspace{1cm} (41)

Inserting values in Eq. 40 we find

\[
\frac{1}{\sqrt{1 - u'^2/c^2}} c = \frac{1}{\sqrt{3/4}} \times \frac{1}{\sqrt{3/4}} c = \frac{1}{3/4} c
\]  \hspace{1cm} (42)

or, simplifying the common factor \( c \), taking the reciprocal of both sides of the equation, and squaring

\[
1 - u'^2/c^2 = 9/16
\]  \hspace{1cm} (43)
with the solution \( u'^2/c^2 = 1 - 9/16 = 7/16 \), or
\[
\frac{u'}{4} = \frac{\sqrt{7}}{4} c
\]  
(44)

Problem 3

A \( \pi^0 \) meson moving with speed \( v \) in the \( x \) direction decays into two photons. One of the photons travels in the \( x \) direction and the other in the minus \( x \) direction.

(a) If one of the photons has an energy that is nine times that of the other photon, calculate the speed of the \( \pi^0 \) meson.

(b) If the speed of the \( \pi^0 \) meson is \( c/2 \), determine the energies of the two photons.

*Please note:* the speed of the \( \pi^0 \) meson in part (b) is not the same as in part (a). In part (b) the speed of the pion is given as \( c/2 \) so that, for the change from the frame where the pion is moving to the rest frame of the pion, \( \beta = (c/2)/c = 1/2 \). In part (a) \( \beta = v/c \) is to be determined.

(This is Problem 4-26 in the textbook, given as problem 3 in assignment 6, with \( z \) direction replaced by \( x \) direction.)

Solution

We simplify the notation by denoting the mass of the \( \pi^0 \) meson, \( m_{\pi^0} = 135 \text{MeV}/c^2 \), simply by \( m \).

(a) Let \( E \) be the energy of each photon in the pion rest frame. The components of its four-momentum vector are given by \( P_0 = E/c, P_1 = \pm E/c, P_2 = P_3 = 0 \). The \( P_0' \) components of the photons four-momenta in the frame where the pion is moving are given by
\[
P_0' = \gamma P_0 - \beta \gamma P_1 \tag{45}
\]
The energies of the two photons, using now the suffixes 1 and 2 to distinguish them, will therefore be given by
\[
E_1' = \gamma E - \beta \gamma E \tag{46}
\]
\[
E_2' = \gamma E + \beta \gamma E \tag{47}
\]
We have
\[ E_2' = 9E_1' \]  
(48)

hence
\[ 1 - \beta = 9(1 + \beta) \]  
(49)

and
\[ \beta = \frac{4}{5} \]  
(50)

(b) The energy of each photon in the rest frame of the pion is \( E = mc^2/2 \).

From \( \beta = 1/2 \) we get \( \gamma = 1/\sqrt{1 - \beta^2} = 1/\sqrt{1 - 1/4} = 1/\sqrt{3/4} \). Substituting into Eqs. 46 and 46 we find now

\[
E_1' = \frac{mc^2}{2\sqrt{3/4}} - \frac{mc^2}{4\sqrt{3/4}} = 135 \text{ MeV} \left( \frac{1}{\sqrt{3}} - \frac{1}{2\sqrt{3}} \right) = 39 \text{ MeV} \]  
(51)

\[
E_1' = \frac{mc^2}{2\sqrt{3/4}} + \frac{mc^2}{4\sqrt{3/4}} = 135 \text{ MeV} \left( \frac{1}{\sqrt{3}} + \frac{1}{2\sqrt{3}} \right) = 117 \text{ MeV} \]  
(52)