Figure 1: Finite square well potential, given by equation 1. The question asks to consider a well of width 2d. This picture shows a well of width 2a.

So far you have studied a quantum mechanical particle in the infinite square well. In this problem, extending off of your previous experience, the goal is to find the bound state \((E < V_0)\) energy value(s) of a quantum mechanical particle of mass \(m\) in the finite square well of height \(V_0\) and width 2\(d\). This potential is given by equation 1 and shown in figure 1. In particular, consider only wave functions symmetric about the wells center \((\psi(x) = \psi(-x))\). You will find that the bound state energies cannot be found analytically. Instead, you want to derive a transcendental equation and show how the energy values can be found graphically.

\[
V(x) = \begin{cases} 
V_0, & x < -d \\
0, & -d < x < d \\
V_0, & x > d
\end{cases}
\] (1)

**a)** Write down the time-independent Shrödinger equation for all three regions \((x < -d, -d \leq x \leq d, \text{and } x > d)\).

**b)** Find the general solution of the Shrödinger equation in each region.

**c)** Use the following constraints to solve for some of the constants in the general solution and to then derive a transcendental equation that can be used to find the energy spectrum.

- \(\lim_{x \to \pm \infty} \psi(x) = 0\)
- \(\psi(x) = \psi(-x)\)
- \(\psi(x)\) is continuous: \(\lim_{x \to \pm d^-} \psi(x) = \lim_{x \to \pm d^+} \psi(x)\)
- \(\psi(x)\) is differentiable: \(\lim_{x \to \pm d^-} \frac{d\psi(x)}{dx} = \lim_{x \to \pm d^+} \frac{d\psi(x)}{dx}\)

**d)** Graphically show – sketch – how one can find the energy value(s) through the transcendental equation derived in part c.
Our potential is given by:

$$V(x) = \begin{cases} 
V_0, & x < -d \\
0, & -d < x < d \\
V_0, & x > d 
\end{cases}$$

The potential has three regions which we will treat separately, and then connect these regions via boundary conditions:

- **Region I**: $x < -d$ : $V(x) = V_0$

  → Shrödinger equation: $-\frac{\hbar^2}{2m} \frac{d^2\psi_1(x)}{dx^2} + V_0 \psi_1(x) = E \psi_1(x) \Rightarrow \frac{d^2\psi_1(x)}{dx^2} = \frac{2m}{\hbar^2} (V_0 - E) \psi_1(x)$

  noting that $V_0 > E$ : $\psi_1(x) = A e^{Bx} + B e^{-Bx} : A$ and $B$ are constants

  \[ q = \frac{\sqrt{2m(V_0 - E)}}{\hbar} \]

- **Region II**: $-d < x < d$ : $V(x) = 0$

  → Shrödinger equation: $-\frac{\hbar^2}{2m} \frac{d^2\psi_2(x)}{dx^2} = E \psi_2(x) \Rightarrow \frac{d^2\psi_2(x)}{dx^2} = -\frac{2mE}{\hbar^2} \psi_2(x)$

  $\psi_2(x) = C \cos(kx) + D \sin(kx) : C$ and $D$ are constants

  \[ k = \frac{\sqrt{2mE}}{\hbar} \]

- **Region III**: $x > d$ : $V(x) = V_0$ not to be confused with energy.

  → Like in Region I : $\psi_3(x) = A e^{Bx} + B e^{-Bx} \hat{E}$ and $F$ and constants, not necessarily the same as $A$ and $B$. 
Combing these results:

\[
\psi(x) = \begin{cases} 
  A e^{ix} + B e^{-ix}, & x < -d \\
  C \cos(kx) + D \sin(kx), & -d < x < d \\
  \tilde{E} e^{i\tilde{x}} + F e^{-i\tilde{x}}, & x > d
\end{cases}
\]

Looks messy! We can simplify this by noting two requirements from \( \psi(x) \):

1. \( \lim_{x \to \pm \infty} \psi(x) = 0 \).

Why?

Recall that \( |\psi(x)|^2 = \psi(x) \psi^*(x) \) is interpreted as a probability density function. That means that it must be normalizable, i.e., we can normalize it by multiplying it by some \( N = \frac{1}{\int_{-\infty}^{\infty} |\psi(x)|^2 dx} \).

However, if \( \lim_{x \to \pm \infty} \psi(x) \neq 0 \), then \( \int_{-\infty}^{\infty} |\psi(x)|^2 dx \to \infty \). This would cause \( N = 0 \) and thus the only normalized wave function would be \( \psi(x) = 0 \), which is a trivial solution to Schrödinger's equation.

Consequence:

\[
\begin{align*}
\lim_{x \to \infty} \psi(x) &= \lim_{x \to \infty} \left( \tilde{E} e^{i\tilde{x}} + F e^{-i\tilde{x}} \right) = 0 \Rightarrow \tilde{E} = 0 \quad \text{or else } \psi(x) \to 0. \\
\lim_{x \to -\infty} \psi(x) &= \lim_{x \to -\infty} \left( A e^{ix} + B e^{-ix} \right) = 0 \Rightarrow B = 0
\end{align*}
\]
\[ \psi(x) = \psi(-x) \]

Why?

Our potential has a very nice symmetry. Namely \( V(x) = V(-x) \). This then gives the Schrödinger a symmetry: It does not change under \( x \rightarrow -x \). This then restricts our possible wave functions by requiring they have a definite symmetry: \( \psi(x) = \psi(-x) \) or \( \psi(x) = -\psi(-x) \). In the question we were asked to consider the even case \( \psi(x) = -\psi(-x) \).

Consequence

After requirement 1, our wave function looks like:

\[
\psi(x) = \begin{cases} 
A e^{8x}, & x < -d \\
C \cos(kx) + D \sin(kx), & -d < x < d \\
F e^{-8x}, & x > d
\end{cases}
\]

Let us consider \( x \rightarrow -x \) for \( x < -d \), \( x > d \) and then \( -d < x < d \):

\[ |x| > d: \]

\[
\frac{A e^{8x}}{x < -d} \rightarrow A e^{-8x} = F e^{-8x} \Rightarrow A = F
\]

\[ |x| < d: \]

\[ C \cos(kx) + D \sin(kx) \rightarrow C \cos(kx) - D \sin(kx) = C \cos(kx) + D \sin(kx) \]

\[ D = 0 \]
Now, our wave function reads:

\[ \Psi(x) = \begin{cases} A e^{i \chi}, & x < -d \\ C \cos(Kx), & -d \leq x \leq d \\ A e^{-i \chi}, & x > d \end{cases} \]

Recall that our goal was to find the energy level(s) \( E \).

Both \( g \) and \( K \) are functions of \( E \), so we can use the wave functions continuity and differentiability:

Let us consider the boundary condition \( \Psi_\Pi(x=d) = \Psi_\Pi(x=d) \) and \( \frac{d}{dx} \Psi_\Pi(x=d) = \frac{d}{dx} \Psi_\Pi(x=d) \). We could consider \( x = -d \), but as \( \Psi(x) = \Psi(-x) \)

We will get the same equations:

**Continuity:** \( \Psi_\Pi(x=d) = \Psi_\Pi(x=d) \Rightarrow C \cos(Kd) = A e^{-g d} \) \( \quad \text{--- (1)} \)

**Differentiability:** \( \frac{d}{dx} \Psi_\Pi(x=d) = \frac{d}{dx} \Psi_\Pi(x=d) \Rightarrow -K C \sin(Kd) = -g A e^{-g d} \) \( \quad \text{--- (2)} \)

Let us divide equation (2) by equation (1):

\[ \frac{2}{1} \Rightarrow \frac{-K C \sin(Kd)}{C \cos(Kd)} = \frac{-g A e^{-g d}}{A e^{-g d}} \quad \rightarrow \quad K \tan(Kd) = g \]
Plugging \( k \) and \( g \) into this equality gives:

\[
\frac{\sqrt{2mE}}{k} \tan \left( \frac{d \sqrt{2mE}}{k} \right) = \frac{\sqrt{2m(V_0 - E)}}{k}.
\]

Simplify:

\[
\tan \left( \frac{d \sqrt{2mE}}{k} \right) = \sqrt{\frac{V_0}{E} - 1}.
\]

Transcendental equation → no analytic form for \( E \).

Because the energy cannot be solved analytically, we can instead sketch the left hand side and right hand side and the points at which they intersect will give the value of \( E \). This can be done carefully and precisely using a change of variable in the textbook (Equations 7.77 and 7.78). However, we can still give it a good sketch.

→ Right hand side: Note that \( \sqrt{\frac{V_0}{E} - 1} \) goes to \( \infty \) as \( E \to 0 \) and goes to zero as \( E \to V_0 \). Thus:

\[
\sqrt{\frac{V_0}{E}} - 1
\]
\[ \text{Left hand side:} \]

\[ \tan \left( \frac{\hbar}{\kappa} \sqrt{2mE} \right) \rightarrow \text{will go to zero at } E = 0, \text{ or more generally } \frac{\partial \sqrt{2mE}}{\kappa} = n \pi \]

\[ \text{where } n \text{ is any integer } \geq 0 \]

\[ \rightarrow \text{will go to } \infty \text{ at } \frac{\partial \sqrt{2mE}}{\kappa} = \frac{\pi}{2} \text{ then } n \text{ is an integer } \geq 0. \]

\[ \text{Hard to tell how many zeros and } \infty \text{ there will be for } 0 < E < V \]

\[ \rightarrow \text{LHS looks like } \tan \left( \frac{\partial \sqrt{2mE}}{\kappa} \right) \]

\[ E \]

Like I said above, it's hard to know how many zeros and \( \infty \)

\[ \tan \left( \frac{\partial \sqrt{2mE}}{\kappa} \right) \] will have \( 0 < E < V_0 \). However, a rough sketch may give:

[Diagram showing energy levels and transitions]

- Bound energy 

Valves.

See Figure 7-13 for sketch given the previously read change of volume.