**Problem 1**

In class you have started to learn about the importance of *eigenvalue problems* in quantum mechanics. Eigenvalue problems come up all the time in physics. In this question we will practice finding eigenvalues and eigenvectors through the classical system shown in figure 1.

![Figure 1: Set up for problem 1](image)

In the set up, we have two carts coupled by Hookean springs (springs that obey Hooke’s law). Writing down Newton’s second law for cart one and cart two, we get the following system of equations:

\[
\begin{align*}
    m_1 \ddot{x}_1 &= -(k_1 + k_2) x_1 + k_2 x_2 \\
    m_2 \ddot{x}_2 &= k_2 x_1 - (k_2 + k_3) x_2
\end{align*}
\]

where \( \ddot{x}_i \) denotes the second time derivative of position - the acceleration - of cart \( i \). Assuming that the the carts and springs are all the same, we have \( m_1 = m_2 = m \) and \( k_1 = k_2 = k_3 = k \). With this, we can write out equations of motion as

\[
\begin{align*}
    \ddot{x}_1 &= -2 \omega^2 x_1 + \frac{k}{m} x_2 \\
    \ddot{x}_2 &= \frac{k}{m} x_1 - 2 \omega^2 x_2
\end{align*}
\]

Denoting \( \omega^2 = k/m \), we then rewrite this in a matrix notation:

\[
\begin{pmatrix}
    \ddot{x}_1 \\
    \ddot{x}_2
\end{pmatrix} =
\begin{pmatrix}
    -2 \omega^2 & \omega^2 \\
    \omega^2 & -2 \omega^2
\end{pmatrix}
\begin{pmatrix}
    x_1 \\
    x_2
\end{pmatrix}
\]

notationally, we will let

\[
A =
\begin{pmatrix}
    -2 \omega^2 & \omega^2 \\
    \omega^2 & -2 \omega^2
\end{pmatrix}
\]

**a)** Find the eigenvalues of matrix \( A \).

**b)** Find the eigenvectors of matrix \( A \).
Problem 3

(Rohlf Example 5-4) From the Bohr model, estimate the wavelength of an electron in the ground state of the hydrogen atom:

*Hint: Recall that the speed of an electron in the ground state is given by $v = \alpha c$ (Eqn. 3.95)*
Problem 4

Given the following wavefunction:

\[ \psi(x) = \begin{cases} 
0 & |x| > L \\
C \cos \left( \frac{x\pi}{2L} \right) & |x| \leq L 
\end{cases} \]

a) Find the normalizing constant \( C \).

Hint 1: Recall that \( |\psi(x)|^2 = \psi^*(x)\psi(x) \) is interpreted as a probability density function.

Hint 2: Recall that \( \cos(2\theta) = 2\cos^2(\theta) - 1 \).

b) Find the expectation value of position, \( \langle x \rangle \).

Hint: Recall the definition of the expectation value of some observable \( \hat{O} \) is given by \( \langle O \rangle = \int_{-\infty}^{\infty} \psi^*(x)\hat{O}\psi(x)dx \).

c) Find the expectation value of momentum, \( \langle p \rangle \).

Hint: Recall that the momentum operator is \( \hat{p} = -i\hbar \frac{d}{dx} \).
Given \( A = \begin{pmatrix} -w^2 & w^2 \\ w^2 & -2w^2 \end{pmatrix} \), we find the eigenvalues by considering the "secular equation": \( \det(A - \lambda I) = 0 \)

\[
\Rightarrow \det\left( \begin{array}{cc} -2w^2 - \lambda & w^2 \\ w^2 & -2w^2 - \lambda \end{array} \right) = (2w^2 + \lambda)^2 - w^4 = 4w^4 + 4\lambda w^2 + \lambda^2 - w^4 = 0
\]

\[ \lambda^2 + 4\lambda w^2 + 3w^4 = 0 \rightarrow \text{expect up to two } \lambda \text{'s b/c 2nd order polynomial,} \]

(Solving for \( \lambda \)’s:)

\[
\lambda = \frac{-4w^2 \pm \sqrt{16w^4 - 12w^4}}{2}
\]

\[ = \left( -4 \pm 2 \right) w^2 = \begin{cases} & -1 \cdot w^2 \\ & -3 \cdot w^2 \end{cases} \]

Therefore: \( \lambda_1 = -w^2, \quad \lambda_2 = -3w^2 \)

b) Find Eigenvectors:

1st)

\[ A \vec{v}_1 = \lambda_1 \vec{v}_1 \Rightarrow \begin{pmatrix} -w^2 & w^2 \\ w^2 & -2w^2 \end{pmatrix} \begin{pmatrix} v_{1x} \\ v_{1y} \end{pmatrix} = -w^2 \begin{pmatrix} v_{1x} \\ v_{1y} \end{pmatrix} \]

\[
\Rightarrow \begin{cases} -2v_{1x} + v_{1y} = -v_{1x} \\ v_{1x} - 2v_{1y} = -v_{1y} \end{cases}
\]

\[
\Rightarrow \begin{cases} v_{1x} = 0 \\ v_{1y} = 0 \end{cases}
\]

Therefore, \( \vec{v}_1 = \vec{0} \)
There is not one single unique eigenvector, so we have room to choose. Let's set $v_{1x} = 1$.

Then, from first constraint: $-2 + v_{1y} = -1 \Rightarrow v_{1y} = 3$

$\vec{v}_1 = \begin{pmatrix} 1 \\ 3 \end{pmatrix}$

2nd

$A\vec{v}_1 = \lambda_1 \vec{v}_1 \Rightarrow -2v_{2x} + v_{2y} = -3v_{2x}$

Letting $v_{2x} = 1 \Rightarrow -2 + v_{2y} = -3 \Rightarrow v_{2y} = -1$

$\vec{v}_2 = \begin{pmatrix} -1 \\ -1 \end{pmatrix}$
Problem 3

Given \( \nu = \alpha c = \frac{\nu}{137} \), we are quite nonrelativistic, and can thus write momentum as:

\[ p = m\nu = ma c. \]

From de Broglie wavelength:

\[ \lambda = \frac{h}{p} = \frac{h}{ma c}, \]

\[ = \frac{hc}{mc^2} a, \]

\[ = \frac{1240 \text{ eV nm}}{(\frac{1}{137}) (5.11 \times 10^6 \text{ eV})} \]

\[ = 0.3 \text{ nm} \]
Problem 4

a)

Normalization Condition:

\[ \int_{-\infty}^{\infty} |\psi(x)|^2 \, dx = 1 \]

\[ \int_{-\infty}^{\infty} |\psi(x)|^2 \, dx = \int_{-L}^{-L} |\psi(x)|^2 \, dx + \int_{-L}^{L} |\psi(x)|^2 \, dx + \int_{L}^{\infty} |\psi(x)|^2 \, dx \]

\[ = \int_{-L}^{-L} 0 \, dx + \int_{-L}^{L} c \cdot \cos \left( \frac{x \pi}{2L} \right)^2 \, dx + \int_{L}^{\infty} 0 \, dx \]

\[ = c^2 \int_{-L}^{L} \cos^2 \left( \frac{x \pi}{2L} \right) \, dx \]

\[ = 2c^2 \int_{0}^{L} \cos^2 \left( \frac{x \pi}{2L} \right) \, dx \]

\[ = \frac{2c^2}{2} \left[ \cos \left( \frac{x \pi}{2L} \right) + 1 \right]_0^L \]

\[ = \frac{2c^2}{2} \left( \frac{L}{\pi} \sin \left( \frac{x \pi}{2L} \right) + X \right) \bigg|_0^L \]

\[ = \frac{2Lc^2}{2} = 1 \quad \Rightarrow \quad c^2 = \frac{1}{L} \quad \rightarrow \quad c = \sqrt{\frac{1}{L}} \]
\[ \langle X \rangle = \int_{-\infty}^{\infty} \psi^* \hat{X} \psi \, dx \]

\[ = \int_{-L}^{L} x \left( \frac{1}{L} \right) \cos^3 \left( \frac{x \pi}{2L} \right) \, dx \]

\[ = \frac{1}{L} \int_{-L}^{L} x \cos^2 \left( \frac{x \pi}{2L} \right) \, dx \]

Integrand is an odd function over symmetric bounds. Thus, the integral is \( \text{zero} \).

\[ \langle X \rangle = 0 \]

\[ \langle p \rangle = \int_{-\infty}^{\infty} \psi^* \hat{p} \psi \, dx = -\frac{i \hbar}{L} \int_{-L}^{L} \cos \left( \frac{x \pi}{2L} \right) \, dx \left[ \cos \left( \frac{x \pi}{2L} \right) \right] \, dx \]

\[ = -\frac{i \hbar}{L} \left[ \cos \left( \frac{x \pi}{2L} \right) \right]_{-L}^{L} + \frac{i \hbar}{L} \int_{-L}^{L} \cos \left( \frac{x \pi}{2L} \right) \cos \left( \frac{x \pi}{2L} \right) \, dx \]

\[ = \frac{i \hbar}{L} \int_{-L}^{L} dx \left[ \cos \left( \frac{x \pi}{2L} \right) \cos \left( \frac{x \pi}{2L} \right) \right] \, dx = -\langle p \rangle \]

\[ \langle p \rangle = -\langle p \rangle \Rightarrow \langle p \rangle = 0 \]