

PY 251 - Principles of Physics

Lecture notes - November 1, 2005.

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The pendulum.

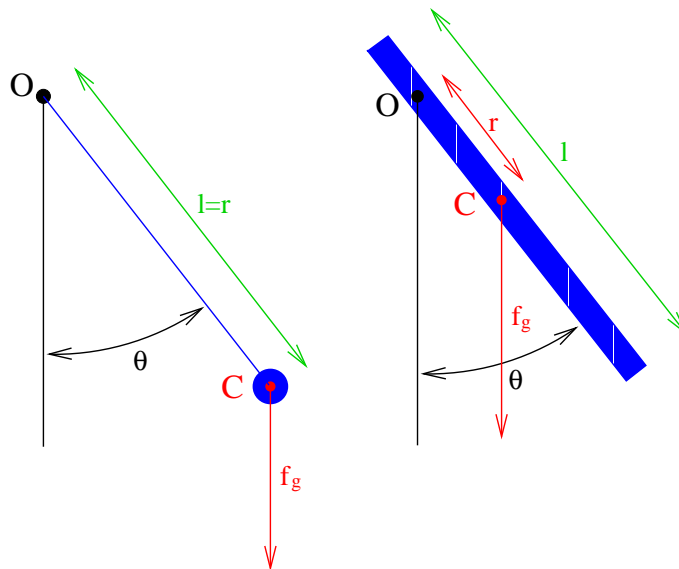


Figure 1: Simple pendulum (left) and physical pendulum (right).

We study the motion of a pendulum, i.e. a solid object which swings without friction around a pivot O under the action of the force of gravity. Such object could be a point-like object tied to O by an inextensible string of negligible mass, as shown in the diagram at left in Fig. 1, or a rigid object of arbitrary shape bound to the pivot through a thin axle around which it can rotate freely, as exemplified in the diagram at right in the same figure. The former (as in the diagram at left) is called a “simple pendulum”, the latter (as in the diagram at right) is called a “physical pendulum”.

We imagine that the pendulum has mass M , length l and moment of inertia I' about its center of mass C , relative to an axis of rotation parallel to the axis of rotation around the pivot. We denote by r the distance of C from O and by θ the angle that the line joining O and C forms with the vertical. θ is

the angle of oscillation of the pendulum and we will assume that θ is small, so that $\sin\theta$ can be well approximated by θ .

The equation of motion for the angle θ is

$$I \frac{d^2\theta}{dt^2} = \tau \quad (1)$$

where I is the moment of inertia about the axis of rotation through O and τ is the magnitude of the torque of the gravitational force \vec{f}_g . τ is given by

$$\tau = -Mgr \sin\theta \quad (2)$$

which, by our assumption of small oscillations, can be well approximated by

$$\tau = -Mgr\theta \quad (3)$$

On the other hand, I can be expressed in terms of I' by

$$I = I' + Mr^2 \quad (4)$$

The value of I' will of course depend on the shape and mass of the pendulum. However, under the assumption that the overall dimension of the pendulum is specified by some length l , I' will be of the form

$$I' = \alpha Ml^2 \quad (5)$$

where α is a constant which depends on the shape of the object, but not on l or M . For example, for the rod shown in the diagram at the right of Fig. 1

$$I' = \frac{1}{12} Ml^2 \quad (6)$$

so that $\alpha = 1/12$. Substituting Eqs. 4, 5 and 3 into Eq. 1 we get

$$(Mr^2 + \alpha Ml^2) \frac{d^2\theta}{dt^2} = -Mgr\theta \quad (7)$$

or

$$\frac{d^2\theta}{dt^2} = -\frac{g}{r + \alpha l^2/r} \theta \quad (8)$$

But this is an equation of the type

$$\frac{d^2\theta}{dt^2} = -\omega^2\theta \quad (9)$$

whose solution, as we know well, is the harmonic motion

$$\theta(t) = A \sin(\omega t + \phi) \quad (10)$$

We conclude that, for small oscillation, the pendulum will move of harmonic motion with

$$\omega = \sqrt{\frac{g}{r + \alpha l^2/r}} \quad (11)$$

and, therefore, period

$$T = 2\pi \sqrt{\frac{r + \alpha l^2/r}{g}} \quad (12)$$

For example, in the case of the simple pendulum, where $r = l$ and $I' = 0$ (hence $\alpha = 0$) we have

$$T = 2\pi \sqrt{\frac{l}{g}} \quad (13)$$

In the case of the rod with the pivot placed at one end we would have $r = l/2$ and thus, with $\alpha = 1/12$,

$$T = 2\pi \sqrt{\frac{2l}{3g}} \quad (14)$$

This shows, in particular, that the period of oscillation of the rod will be the same as the period of oscillation of a simple pendulum of $2/3$ the length of the rod.

Damped oscillations.

We study the motion along the x axis of an object of mass m subject to an elastic force of magnitude $f_e = -kx$ and to a damping force of magnitude proportional to the velocity of the object $f_d = -bv = -b dx/dt$. The equation of motion is

$$ma = m \frac{d^2x}{dt^2} = f_e + f_d = -kx - b \frac{dx}{dt} \quad (15)$$

or

$$m \frac{d^2x}{dt^2} + b \frac{dx}{dt} + kx = 0 \quad (16)$$

Since the exponential function reproduces itself under derivation [$d \exp(\alpha t)/dt = \alpha \exp(\alpha t)$], we try a solution of the form

$$x(t) = Ae^{\alpha t} \quad (17)$$

Carrying out the differentiations, we get

$$(m\alpha^2 + b\alpha + k)Ae^{\alpha t} = 0 \quad (18)$$

which will be solved if

$$m\alpha^2 + b\alpha + k = 0 \quad (19)$$

Equation 19 has the two solutions

$$\alpha_{1,2} = -\frac{b}{2m} \pm \sqrt{\left(\frac{b}{2m}\right)^2 - \frac{k}{m}} \quad (20)$$

If $b/(2m) > \sqrt{k/m}$ this solves our problem. Both α_1 and α_2 are negative real numbers and the most general solution to the equation of motion can be written as

$$x(t) = A_1 e^{\alpha_1 t} + A_2 e^{\alpha_2 t} = A_1 e^{-|\alpha_1|t} + A_2 e^{-|\alpha_2|t} \quad (21)$$

where the constants A_1 and A_2 can be chosen in such a way as to satisfy the initial conditions for $x(0)$ and $v(0)$. The motion described by Eq. 21 is the superposition of two decreasing exponentials. It is a strongly damped motion. Two examples of such motion are given in Fig. 2: in the graph at the top $v(0) = 0$, while in the graph at the bottom $x(0) = 0$.

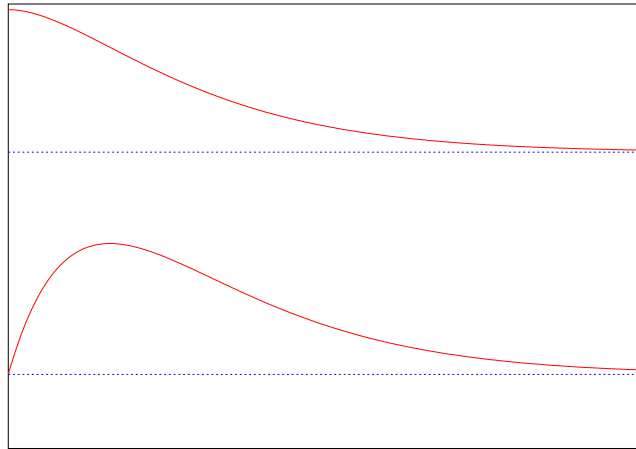


Figure 2: Damped oscillatory motion: strong damping.

If $b/(2m) < \sqrt{k/m}$, however, the solutions of Eq. 19 turn out to be complex numbers, with α_2 the complex conjugate of α_1 :

$$\alpha_{1,2} = -\frac{b}{2m} \pm i\sqrt{\frac{k}{m} - \left(\frac{b}{2m}\right)^2} \quad (22)$$

It is convenient to introduce the notation

$$\gamma = \frac{b}{2m} \quad (23)$$

$$\omega_0^2 = \frac{k}{m} \quad (24)$$

$$\omega'^2 = \frac{k}{m} - \left(\frac{b}{2m}\right)^2 = \omega_0^2 - \gamma^2 \quad (25)$$

With this

$$\alpha_{1,2} = -\gamma \pm i\omega' \quad (26)$$

We get a clue on how to proceed in this case from the theory of complex numbers, which tells us that

$$e^{i\phi} = \cos \phi + i \sin \phi \quad (27)$$

This prompts us to look for a solution of the form

$$x(t) = Ae^{-\gamma t} \cos(\omega' t) \quad (28)$$

Taking the derivatives we get

$$\frac{dx}{dt} = -\gamma Ae^{-\gamma t} \cos(\omega' t) - \omega' Ae^{-\gamma t} \sin(\omega' t) \quad (29)$$

and

$$\frac{d^2x}{dt^2} = \gamma^2 Ae^{-\gamma t} \cos(\omega' t) + 2\gamma\omega' Ae^{-\gamma t} \sin(\omega' t) - \omega'^2 Ae^{-\gamma t} \cos(\omega' t) \quad (30)$$

Substituting into the equation of motion we get

$$\begin{aligned} m\gamma^2 Ae^{-\gamma t} \cos(\omega' t) + 2m\gamma\omega' Ae^{-\gamma t} \sin(\omega' t) - m\omega'^2 Ae^{-\gamma t} \cos(\omega' t) \\ - b\gamma Ae^{-\gamma t} \cos(\omega' t) - b\omega' Ae^{-\gamma t} \sin(\omega' t) \\ + kAe^{-\gamma t} \cos(\omega' t) = 0 \end{aligned} \quad (31)$$

simplifying the common factor $A \exp(-\gamma t)$ and equating the coefficients of the $\cos(\omega't)$ and $\sin(\omega't)$ terms separately, we get

$$m\gamma^2 - m\omega'^2 - b\gamma + k = 0 \quad (32)$$

$$2m\gamma\omega' - b\omega' = 0 \quad (33)$$

Using for γ and ω' their values, as given in Eqs. 23 and 25, we find

$$\begin{aligned} m\left(\frac{b}{2m}\right)^2 - m\left[\frac{k}{m} - \left(\frac{b}{2m}\right)^2\right] - b\frac{b}{2m} + k \\ = \frac{b^2}{4m} - k + \frac{b^2}{4m} - \frac{b^2}{2m} + k = 0 \end{aligned} \quad (34)$$

and

$$\left(2m\frac{b}{2m} - b\right)\omega' = (b - b)\omega' = 0 \quad (35)$$

and thus we see that the equation of motion is indeed satisfied.

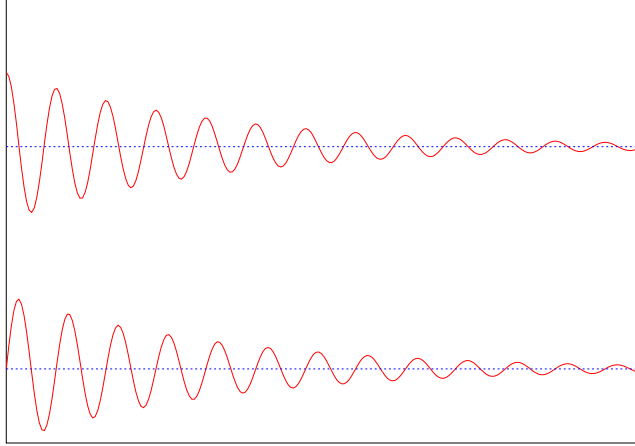


Figure 3: Damped oscillatory motion: weak damping.

Repeating the steps above with

$$x(t) = Ae^{-\gamma t} \sin(\omega't) \quad (36)$$

we see that this also satisfies the equation of motion. We conclude that in the case $b/(2m) < \sqrt{k/m}$ (small to moderate damping) the most general solution to the equation of motion is

$$x(t) = e^{-\gamma t}[A \cos(\omega' t) + B \sin(\omega' t)] \quad (37)$$

or, equivalently,

$$x(t) = e^{-\gamma t} C \cos(\omega' t + \phi) \quad (38)$$

The constants A and B in Eq. 37, or the constant C and the angle ϕ (called the initial phase) in Eq. 38 can be chosen so as to satisfy the initial conditions for $x(0)$ and $v(0)$. The motion is an oscillatory motion with amplitude that decreases exponentially in time. An example of such motion is given in Fig. 2: in the graph at the top $v(0) = 0$, while in the graph at the bottom $x(0) = 0$.

Forced oscillations.

We study the motion of the damped oscillator considered in the previous section in presence of an external oscillatory force $f_e(t) = F \sin(\omega t)$. The external force is also called the driving force. The equation of motion is now

$$ma = m \frac{d^2 x}{dt^2} = f_e + f_d + f_s = -kx - b \frac{dx}{dt} + F \sin(\omega t) \quad (39)$$

or

$$m \frac{d^2 x}{dt^2} + b \frac{dx}{dt} + kx = F \sin(\omega t) \quad (40)$$

Given any two solutions of this equation $x_1(t)$ and $x_2(t)$, their difference satisfies the homogeneous equation $m d^2 x/dt^2 + b dx/dt + kx = 0$, which we studied in the previous section. It follows that the most general solution of Eq. 40 can be written as $x(t) + x_0(t)$, where $x(t)$ is any solution of Eq. 40, while $x_0(t)$ is the most general solution of the homogeneous equation. Since we have already found the most general solution to the homogeneous equation, we can now restrict our attention to finding a particular solution of Eq. 40, which we seek in the form

$$x(t) = A \sin(\omega t - \phi) \quad (41)$$

In other words, we are looking for a particular solution which is an oscillatory function with the same frequency as the driving force, but allowing for an amplitude A and a phase ϕ , to be determined.

Differentiating Eq. 41 we get

$$\frac{dx}{dt} = A\omega \cos(\omega t - \phi) \quad (42)$$

$$\frac{d^2x}{dt^2} = -A\omega^2 \sin(\omega t - \phi) \quad (43)$$

Substituting into Eq. 40 we find

$$-mA\omega^2 \sin(\omega t - \phi) + bA\omega \cos(\omega t - \phi) + kA \sin(\omega t - \phi) = F \sin(\omega t) \quad (44)$$

At this point we could use the addition formulae for sine and cosine to express the trigonometric functions in the l.h.s. of this equation in terms of $\sin(\omega t)$ and $\cos(\omega t)$ and equate the coefficients of the $\sin(\omega t)$ and $\cos(\omega t)$ terms separately. It actually saves a little algebra to write instead

$$F \sin(\omega t) = F \sin(\omega t - \phi + \phi) = F \sin(\omega t - \phi) \cos \phi + F \cos(\omega t - \phi) \sin \phi \quad (45)$$

Substituting into Eq. 44 we get

$$\begin{aligned} -mA\omega^2 \sin(\omega t - \phi) + bA\omega \cos(\omega t - \phi) + kA \sin(\omega t - \phi) \\ = F \sin(\omega t - \phi) \cos \phi + F \cos(\omega t - \phi) \sin \phi \end{aligned} \quad (46)$$

which gives us the two separate conditions

$$-mA\omega^2 + kA = F \cos \phi \quad (47)$$

and

$$bA\omega = F \sin \phi \quad (48)$$

Taking the ratio of Eqs. 48 and 47 gives us

$$\frac{\sin \phi}{\cos \phi} = \tan \phi = \frac{b\omega}{k - m\omega^2} \quad (49)$$

or

$$\phi = \arctan \left(\frac{b\omega}{k - m\omega^2} \right) = \arctan \left(\frac{b\omega}{m(\omega_0^2 - \omega^2)} \right) \quad (50)$$

with $\omega_0 = \sqrt{k/m}$.

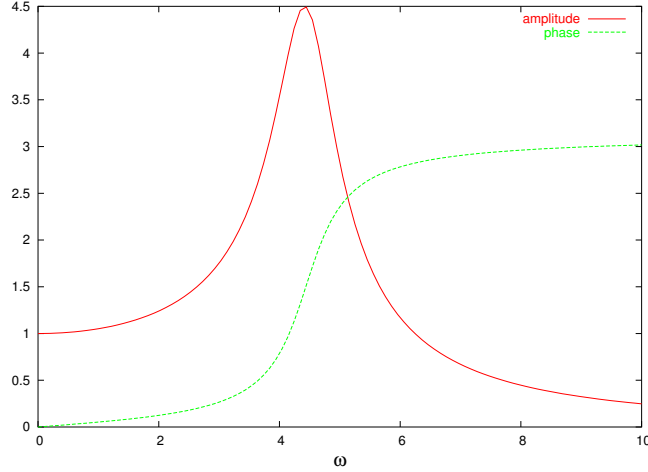


Figure 4: Amplitude and phase of a forced oscillation.

Squaring and summing Eqs. 47 and 48 gives

$$[(m\omega^2 - k)^2 + b^2\omega^2]A^2 = F^2 \quad (51)$$

or

$$A = \frac{F}{\sqrt{(m\omega^2 - k)^2 + b^2\omega^2}} = \frac{F}{\sqrt{m^2(\omega^2 - \omega_0^2)^2 + b^2\omega^2}} \quad (52)$$

Figure 4 illustrates the behavior of the amplitude A and phase ϕ of the forced oscillations. The amplitude of the oscillation exhibits a peak at the resonant frequency $\omega = \sqrt{\omega_0^2 - b^2/(2m^2)}$, while the phase grows from 0 to π as ω varies from 0 to ∞ , taking value $\pi/2$ at $\omega = \omega_0$.

Waves in one dimension.

We consider a system of identical beads of mass m equally spaced at a distance δ on a string of negligible mass subject to a tension T . We denote by x_i, y_i the coordinates of the i^{th} bead (see Fig. 5). We imagine that in the equilibrium position the beads have coordinates

$$x_i = i\delta \quad y_i = 0 \quad (53)$$

We wish to determine the motion of the system when the beads are taken away from their equilibrium positions. We make the assumption that the y

coordinates of the beads stay much smaller than the distance between beads δ . Under this assumption

$$x_{i+1} - x_i = \sqrt{\delta^2 - (y_{i+1} - y_i)^2} = \delta - \frac{(y_{i+1} - y_i)^2}{2\delta} + O[(y_{i+1} - y_i)^4] \approx \delta \quad (54)$$

and we only need to consider the time dependence of the coordinates $y_i(t)$.

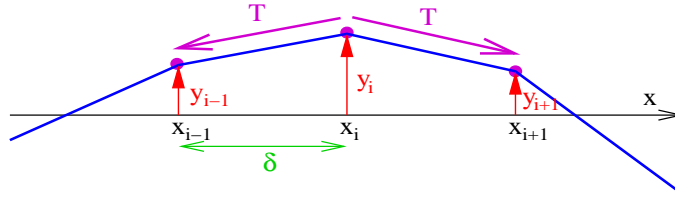


Figure 5: Waves: beads on a string.

The y component of the force that bead $i - 1$ exerts on bead i through the tension of the string is (see Fig. 5)

$$f_{y,i,i-1} = -T \frac{y_i - y_{i-1}}{\delta} \quad (55)$$

Similarly, the y component of the force exerted by bead $i + 1$ on bead i is

$$f_{y,i,i+1} = -T \frac{y_i - y_{i+1}}{\delta} \quad (56)$$

It follows that the equation of motion for $y_i(t)$ is

$$m \frac{d^2 y_i}{dt^2} = f_{y,i,i-1} + f_{y,i,i+1} = -T \frac{y_i - y_{i-1}}{\delta} - T \frac{y_i - y_{i+1}}{\delta} = T \frac{y_{i+1} + y_{i-1} - 2y_i}{\delta} \quad (57)$$

We look for oscillatory solutions of the form

$$y_i(t) = A \sin(kx_i - \omega t + \phi) \quad (58)$$

This gives

$$\begin{aligned} y_{i+1}(t) &= A \sin(kx_{i+1} - \omega t + \phi) = A \sin(k(x_i + \delta) - \omega t + \phi) \\ &= A \sin(kx_i - \omega t + \phi + k\delta) \\ &= A \sin(kx_i - \omega t + \phi) \cos(k\delta) + A \cos(kx_i - \omega t + \phi) \sin(k\delta) \end{aligned} \quad (59)$$

Similarly

$$\begin{aligned}
 y_{i-1}(t) &= A \sin(kx_{i-1} - \omega t + \phi) \\
 &= A \sin(kx_i - \omega t + \phi) \cos(k\delta) - A \cos(kx_i - \omega t + \phi) \sin(k\delta)
 \end{aligned} \tag{60}$$

When we substitute these expressions back into the equation of motion (Eq. 57) the terms with $\sin(k\delta)$ cancel and we are left with

$$\begin{aligned}
 m \frac{d^2 A \sin(kx_i - \omega t + \phi)}{dt^2} &= -mA\omega^2 \sin(kx_i - \omega t + \phi) \\
 &= TA \sin(kx_i - \omega t + \phi) \frac{2 \cos(k\delta) - 2}{\delta}
 \end{aligned} \tag{61}$$

We see that this equation will be satisfied provided that we take

$$\omega = \omega(k) = \sqrt{\frac{2T[1 - \cos(k\delta)]}{m\delta}} \tag{62}$$

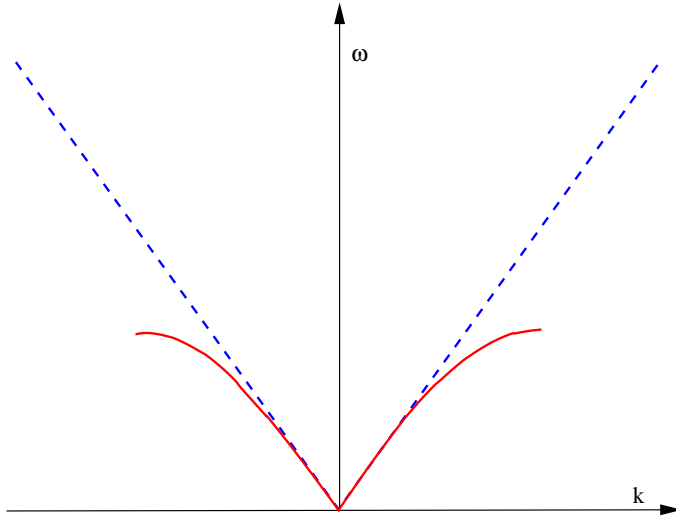


Figure 6: Dispersion formulae.

This relation is called a dispersion formula. The dispersion formula given by Eq. 62 is illustrated by the solid line in Fig. 6. Since k is arbitrary, we conclude that we have an infinity of possible solutions to the equations of motion, each choice of k and ϕ leading to a different solution. We should

however notice that an increase of k by $2\pi/\delta$ does not lead to a different solution, since (remember that $x_i = i\delta$)

$$\begin{aligned} A \sin \left[\left(k + \frac{2\pi}{\delta} \right) x_i - \omega t + \phi \right] &= A \sin \left[kx_i - \omega t + \phi + \frac{2\pi i\delta}{\delta} \right] \\ &= A \sin[kx_i - \omega t + \phi + 2\pi i] = A \sin[kx_i - \omega t + \phi] \end{aligned} \quad (63)$$

For this reason, one should limit the range of k to

$$-\frac{\pi}{\delta} \leq k \leq \frac{\pi}{\delta} \quad (64)$$

Values of k outside of this range do not lead to different solutions. (This range is implicitly shown in Fig. 5, where we see that the solid line ends abruptly at the end of the allowed range of values of k .)

We are interested in going to the continuum limit of a massive string. For the purpose we let the mass of the beads be

$$m = \mu\delta \quad (65)$$

and take the limit $\delta \rightarrow 0$. In this limit the coordinates x_i fill the entire x -axis, the string of beads becomes a continuous string of linear mass density μ and the vertical displacements $y_i(t)$ become a continuous function of x and t : $y_i(t) = y(x_i, t) \rightarrow y(x, t)$.

In order to take the limit $\delta \rightarrow 0$ we first replace m with $\mu\delta$ in Eq. 57 to rewrite the equation of motion in the form

$$\mu \frac{d^2 y}{dt^2} = T \frac{y_{i+1} + y_{i-1} - 2y_i}{\delta^2} \quad (66)$$

We then notice that

$$\begin{aligned} \lim_{\delta \rightarrow 0} \frac{y_{i+1} + y_{i-1} - 2y_i}{\delta^2} &= \lim_{\delta \rightarrow 0} \frac{1}{\delta} \left(\frac{y(x_{i+1}) - y(x_i)}{\delta} - \frac{y(x_i) - y(x_{i-1}))}{\delta} \right) \\ &= \lim_{\delta \rightarrow 0} \frac{1}{\delta} [y'(x_i + \delta/2) - y'(x_i - \delta/2) + O(\delta^2)] = y''(x_i) \end{aligned} \quad (67)$$

Thus the equation of motion becomes

$$\mu \frac{\partial^2 y(x, y)}{\partial t^2} = T \frac{\partial^2 y(x, y)}{\partial x^2} \quad (68)$$

where we have used the partial derivative symbols, since now y is a function of the two variables x and t .

In the limit $\delta \rightarrow 0$ the solution of Eq. 58 becomes

$$y(x, t) = A \sin(kx - \omega t + \phi) \quad (69)$$

which solves the continuous x equations of motion provided that ω and k are related by

$$\omega = \lim_{\delta \rightarrow 0} \sqrt{\frac{2T[1 - \cos(k\delta)]}{\mu\delta^2}} = \sqrt{\frac{T}{\mu}} |k| \quad (70)$$

This dispersion formula is illustrated by the dashed line in Fig. 6. It is actually straightforward to verify directly that, with ω related to k by $\omega = \sqrt{T/\mu} |k|$, the expression of Eq. 69 satisfies Eq. 68.

The equation of motion 68 is called the equation of D'Alembert or the wave equation. The solution given in Eq. 69 represents a sinusoidal wave which propagates with velocity

$$v = \frac{\omega}{k} = \pm \sqrt{\frac{T}{\mu}} \quad (71)$$

ω is related to the frequency of the wave ν by

$$\omega = 2\pi\nu \quad (72)$$

k is called the wave-number of the wave and is related to the wave-length λ by

$$\lambda = \frac{2\pi}{k} \quad (73)$$

Energy of the wave.

The wave will carry both kinetic and potential energy. In order to evaluate the energy of the wave it is useful to go back to the system of oscillating beads. The kinetic energy of bead i is clearly

$$K_i = \frac{m}{2} \left(\frac{dy_i}{dt} \right)^2 \quad (74)$$

Insofar as the potential energy is concerned, the force exerted on bead i by bead $i - 1$, given by (see Eq. 55)

$$f_{y,i,i-1} = -T \frac{y_i - y_{i-1}}{\delta}$$

can be derived from a potential energy

$$U_{i,i-1} = \frac{T}{2} \frac{(y_i - y_{i-1})^2}{\delta} \quad (75)$$

Indeed, we have

$$-\frac{dU_{i,i-1}}{dy_i} = -T \frac{y_i - y_{i-1}}{\delta} = f_{y,i,i-1} \quad (76)$$

Incidentally, $U_{i,i-1}$ also accounts for the force that bead i exerts on bead $i-1$, since

$$-\frac{dU_{i,i-1}}{dy_{i-1}} = -T \frac{y_{i-1} - y_i}{\delta} = f_{y,i-1,i} \quad (77)$$

We conclude that the energy of the system of oscillating beads consists of a kinetic energy term

$$K = \sum_i K_i = \frac{m}{2} \sum_i \left(\frac{dy_i}{dt} \right)^2 \quad (78)$$

and a potential energy term

$$U = \sum_i U_{i,i-1} = \frac{T}{2} \sum_i \frac{(y_i - y_{i-1})^2}{\delta} \quad (79)$$

Note that the potential energy is the sum of terms associated to the pairs of successive beads, or, equivalently, to the intervals between successive beads.

When we go to the continuum limit, K and U take the form

$$K = \lim_{\delta \rightarrow 0} \frac{\mu}{2} \sum_i \left(\frac{dy_i}{dt} \right)^2 \delta = \frac{\mu}{2} \int \left(\frac{\partial y}{\partial t} \right)^2 dx \quad (80)$$

$$U = \lim_{\delta \rightarrow 0} \frac{T}{2} \sum_i \frac{(y_i - y_{i-1})^2}{\delta} = \frac{T}{2} \sum_i \frac{(y_i - y_{i-1})^2}{\delta^2} \delta = \frac{T}{2} \int \left(\frac{\partial y}{\partial x} \right)^2 dx \quad (81)$$

The total energy is the sum of the two

$$E = \int \frac{1}{2} \left[\mu \left(\frac{\partial y}{\partial t} \right)^2 + T \left(\frac{\partial y}{\partial x} \right)^2 \right] dx \quad (82)$$

The integrand in Eq. 82

$$\mathcal{E} = \frac{1}{2} \left[\mu \left(\frac{\partial y}{\partial t} \right)^2 + T \left(\frac{\partial y}{\partial x} \right)^2 \right] \quad (83)$$

is called the “energy density” of the wave. It gives the amount of energy in the system per unit of length. Note that, while the total energy E is conserved, the energy is not locally conserved and so the energy density \mathcal{E} can vary with time. One can derive, however, an interesting equation about the way the local energy content changes. For the purpose, let us consider the amount of energy in an interval $x_0 \leq x \leq x_1$. This is given by

$$\Delta E = \int_{x_0}^{x_1} \frac{1}{2} \left[\mu \left(\frac{\partial y}{\partial t} \right)^2 + T \left(\frac{\partial y}{\partial x} \right)^2 \right] dx \quad (84)$$

Taking the derivative with respect to time we get

$$\begin{aligned} \frac{d\Delta E}{dt} &= \int_{x_0}^{x_1} \frac{1}{2} \frac{\partial}{\partial t} \left[\mu \left(\frac{\partial y}{\partial t} \right)^2 + T \left(\frac{\partial y}{\partial x} \right)^2 \right] dx \\ &= \int_{x_0}^{x_1} \left[\mu \frac{\partial y}{\partial t} \frac{\partial^2 y}{\partial t^2} + T \frac{\partial y}{\partial x} \frac{\partial^2 y}{\partial x \partial t} \right] dx \end{aligned} \quad (85)$$

The last term in the above equation can be integrated by parts:

$$\begin{aligned} \int_{x_0}^{x_1} T \frac{\partial y}{\partial x} \frac{\partial^2 y}{\partial x \partial t} dx &= \int_{x_0}^{x_1} -T \frac{\partial^2 y}{\partial x^2} \frac{\partial y}{\partial t} dx + \left[T \frac{\partial y}{\partial x} \frac{\partial y}{\partial t} \right]_{x_0}^{x_1} \\ &= \int_{x_0}^{x_1} -T \frac{\partial^2 y}{\partial x^2} \frac{\partial y}{\partial t} dx + T \frac{\partial y}{\partial x} \frac{\partial y}{\partial t} \Big|_{x_1} - T \frac{\partial y}{\partial x} \frac{\partial y}{\partial t} \Big|_{x_0} \end{aligned} \quad (86)$$

Putting everything together we get

$$\frac{d\Delta E}{dt} = \int_{x_0}^{x_1} \frac{\partial y}{\partial t} \left[\mu \frac{\partial^2 y}{\partial t^2} - T \frac{\partial^2 y}{\partial x^2} \right] dx + T \frac{\partial y}{\partial x} \frac{\partial y}{\partial t} \Big|_{x_1} - T \frac{\partial y}{\partial x} \frac{\partial y}{\partial t} \Big|_{x_0} \quad (87)$$

But on account of the equation of motion, Eq. 68, the integral in the r.h.s. of the equation above vanishes, and thus we have more simply

$$\frac{d\Delta E}{dt} = T \frac{\partial y}{\partial x} \frac{\partial y}{\partial t} \Big|_{x_1} - T \frac{\partial y}{\partial x} \frac{\partial y}{\partial t} \Big|_{x_0} \quad (88)$$

This equation has a simple interpretation. We can take

$$\Phi(x, t) = -T \frac{\partial y}{\partial x} \frac{\partial y}{\partial t} \quad (89)$$

as the flux of energy flowing along the system. Then Eq. 88 tells us that the derivative with respect to time of the energy of the wave in the interval between x_0 and x_1 is equal to the flux of energy coming into the interval less the flux of energy leaving the interval:

$$\frac{d\Delta E}{dt} = \Phi(x_0, t) - \Phi(x_1, t) \quad (90)$$

In particular, if the wave is totally contained in the interval x_0, x_1 , so that $y(x, t)$ vanishes for $x = x_0$ and $x = x_1$, then the flux of energy also vanishes at x_0 and x_1 and the total energy in x_0, x_1 is conserved.

Finally let us notice that, if we consider a wave like the one in Eq. 69, $y(x, t) = A \sin(kx - \omega t + \phi)$, we have $(\partial y / \partial t)^2 = A^2 \omega^2 \cos^2(kx - \omega t + \phi)$, $(\partial y / \partial x)^2 = A^2 k^2 \cos^2(kx - \omega t + \phi)$ and thus its energy density is

$$\mathcal{E} = \frac{1}{2} A^2 [\mu \omega^2 + T k^2] \cos^2(kx - \omega t + \phi) = \mu \omega^2 A^2 \cos^2(kx - \omega t + \phi) \quad (91)$$

where we have used Eq. 70 to replace $T k^2$ with $\mu \omega^2$. In particular we see that the energy of the wave is proportional to the square of the amplitude of oscillation.

Superposition of waves.

Both Eq. 57, for the oscillations of a string of beads, and Eq. 68, for the oscillations of a continuous string, are linear equations. This means that, if $y^{(1)}$ and $y^{(2)}$ are two solutions to the equation, so is the linear combination $c_1 y^{(1)} + c_2 y^{(2)}$, where c_1 and c_2 are two arbitrary, constant coefficients. This is why the special solutions we found are so important. Indeed, considering for example the equation for the continuous string, one can superimpose the solutions of the form $A \sin(kx - \omega t + \phi)$ into

$$y(x, t) = \int_{-\infty}^{\infty} A(k) \sin[kx - \omega(k)t + \phi(k)] dk \quad (92)$$

where $\omega(k) = \sqrt{T/\mu} |k|$, and one can show that this is the most general solution to the wave equation, Eq. 68.

As a parenthetical remark, we observe that Eq. 92 can be rewritten in the form

$$y(x, t) = \int_{-\infty}^0 A(k) \sin[kx - \omega(k)t + \phi(k)] dk$$

$$\begin{aligned}
& + \int_0^\infty A(k) \sin[kx - \omega(k)t + \phi(k)] dk \\
& = \int_{-\infty}^0 A(k) \sin[k(x + vt) + \phi(k)] dk \\
& + \int_0^\infty A(k) \sin[k(x - vt) + \phi(k)] dk
\end{aligned} \tag{93}$$

where we have used $\omega = v|k| = -vk$ for $k < 0$, $\omega = vk$ for $k > 0$. But the integral $\int_{-\infty}^0 A(k) \sin[k(x + vt) + \phi(k)] dk$ is obviously some function $F(x + vt)$ of the combination of variables $x + vt$, while the integral $\int_0^\infty A(k) \sin[k(x - vt) + \phi(k)] dk$ is some function $G(x - vt)$ of $x - vt$. Thus we conclude that the most general solution to Eq. 68 is the superposition of an arbitrary function of $x + vt$ (a wave traveling to the left with velocity $-v$) and an arbitrary function of $x - vt$ (a wave traveling to the right with velocity v). However this is special to the case of the wave equation in one space dimension.

We return to the superposition of waves with two examples.

In the first example we use the superposition to find the most general solution to the wave equation subject to the boundary conditions that the displacement $y(x, t)$ vanishes for $x = 0$ and $x = L$, i.e. we study the motion of a string of length L fixed at the end points. We note that the superposition of waves

$$y(x, t) = A[\sin(kx - \omega(k)t + \phi) - \sin(-kx - \omega(-k)t + \phi)] \tag{94}$$

can be rewritten in the form

$$\begin{aligned}
y(x, t) & = A[\sin(kx) \cos(\omega(k)t - \phi) - \cos(kx) \sin(\omega(k)t - \phi) \\
& + \sin(kx) \cos(\omega(k)t - \phi) + \cos(kx) \sin(\omega(k)t - \phi)] \\
& = 2A \sin(kx) \cos(\omega(k)t - \phi)
\end{aligned} \tag{95}$$

(we have used the addition formulae for the trigonometric functions as well as the fact that $\omega(-k) = \omega(k)$). It is clear from the final expression that $y(x, t)$ vanishes for $x = 0$, and thus satisfies one of the boundary conditions. Imposing the other boundary condition, namely $y(L, t) = 0$, gives us then

$$2A \sin(kL) \cos(\omega(k)t - \phi) = 0 \tag{96}$$

which will be satisfied if and only if k is a multiple of π/L , i.e. $k = \pi n/L$. Thus we find, for the most general solution to the equations of motion of the

string with fixed ends,

$$y(x, t) = \sum_{n=0}^{\infty} A_n \sin(\pi n/L) \cos(\pi nvt/L - \phi_n) = 0 \quad (97)$$

where A_n and ϕ_n are arbitrary amplitudes and phases. Wave of the form of Eq. 96 are called standing waves. The most general solution is thus a superposition of a standing wave of wavelength $2L$ and period $2L/v$ and of all its higher harmonics. Our construction shows that a standing wave can be thought of as the superposition of two waves of equal amplitude and opposite phases traveling in opposite directions.

The second example we consider is the superposition of two waves with different but close wave-numbers which, as we will see, gives origin to beats. We take the two wave-numbers to be

$$k_{\pm} = k_0 \pm \frac{\Delta k}{2} \quad (98)$$

The corresponding ω will be

$$\omega_{\pm} = \omega \left(k_0 \pm \frac{\Delta k}{2} \right) = \omega_0 \pm \frac{\Delta \omega}{2} \quad (99)$$

where we have taken the last identity as the definition of ω_0 and $\Delta \omega$.

We superimpose these two waves with equal amplitude A and, without loss of generality, with vanishing phases, and get

$$y(x, t) = A[\sin(k_+x - \omega_+t) + \sin(k_-x - \omega_-t)] \quad (100)$$

This can be rewritten as

$$y(x, t) = A \left[\sin \left(\frac{(k_+ + k_-)x - (\omega_+ + \omega_-)t}{2} + \frac{(k_+ - k_-)x - (\omega_+ - \omega_-)t}{2} \right) + \sin \left(\frac{(k_+ + k_-)x - (\omega_+ + \omega_-)t}{2} - \frac{(k_+ - k_-)x - (\omega_+ - \omega_-)t}{2} \right) \right] \quad (101)$$

which, using the addition formulae for trigonometric functions, becomes

$$\begin{aligned} & y(x, t) = \\ & = 2A \sin \left(\frac{(k_+ + k_-)x - (\omega_+ + \omega_-)t}{2} \right) \cos \left(\frac{(k_+ - k_-)x - (\omega_+ - \omega_-)t}{2} \right) \\ & = 2A \sin(k_0x - \omega_0t) \cos(\Delta k x - \Delta \omega t) \end{aligned} \quad (102)$$

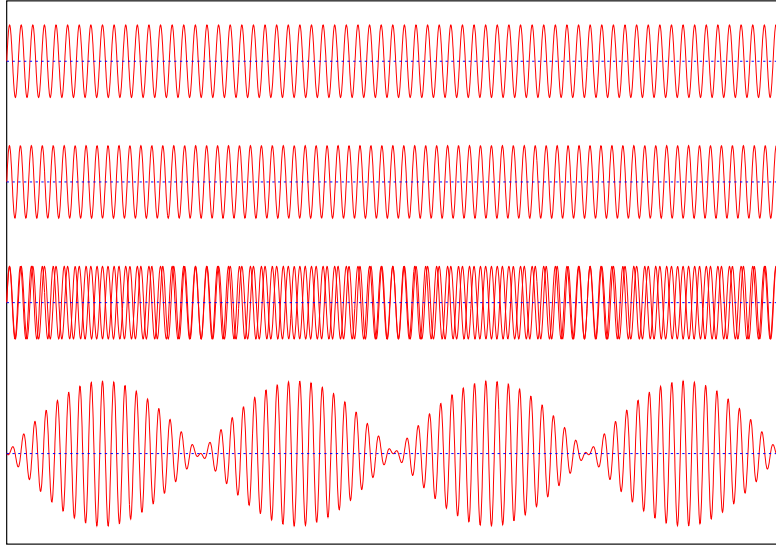


Figure 7: Superposition of waves: beats.

From this last equation we see that the resulting wave has the shape of a sinusoidal wave $\sin(k_0 x - \omega_0 t)$, but with an amplitude $2A \cos(\Delta k x - \Delta \omega t)$ which is modulated in space and time. Since we have taken the wave number Δk and frequency $\Delta \omega/(2\pi)$ of the modulation to be much smaller than k_0 and $\omega_0/(2\pi)$, the modulating wave encompasses many wave lengths of the underlying sinusoidal wave: the superposition of the two waves exhibits “beats” in space and time. This type of superposition is illustrated in Fig. 7, where the two top diagrams show the individual waves which are superimposed, the third diagram from the top shows the two waves drawn on top of each other, and the bottom diagram shows the actual superposition of the two waves.

To conclude, we notice that the modulating profile $\cos(\Delta k x - \Delta \omega t)$ also represents a traveling wave. Its velocity, however, is not $v = \omega_0/k_0$, but rather

$$v_g = \frac{\Delta \omega}{\Delta k} \approx \left. \frac{d\omega(k)}{dk} \right|_{k_0} \quad (103)$$

v_g is called the “group velocity” of the wave packet formed by the superimposed waves, while $v = \omega_0/k_0$ is called the “phase velocity” of the modulated wave. In the particular case of the waves propagating on the continuous

string with dispersion formula $\omega = v|k|$ group velocity and phase velocity coincide, but with a different dispersion formula, for example with the dispersion formula of Eq. 62 for the string with beads, the two would be different.

Reading.

Read Fishbane et al. Chapter 13 pages 366-387, Chapter 14 pages 397-426 and Chapter 15 pages 435-446.