Leapfrog/Verlet method including damping

We assumed velocity-independent force (acceleration) in

$$v_{n+1/2} = v_{n-1/2} + \Delta_t a_n$$

we do not have v_n for $a_n=a(x_n,v_n,t)$

$$x_{n+1} = x_n + \Delta_t v_{n+1/2}$$

We can still use this form, with $a_n \rightarrow a(x_n,v_{n-1/2},t)$, where error is $O(\Delta_t)$

- x error is then $O(\Delta_t^3)$ instead of $O(\Delta_t^4)$ [see by expanding a(v) in v]

To do better, first separate out dissipative part of force:

$$a(x,v,t) = \frac{1}{m}[F(x,t) - G(v)]$$

Consider the approximation

$$a(x_n,v_n,t_n) \approx [F(x_n,t_n) - G(v_{n-1/2})]/m$$

and use this for intermediate (^) velocity and position:

$$\hat{v}_{n+1/2} = v_{n-1/2} + \Delta_t[F(x_n,t_n) - G(v_{n-1/2})]/m$$

$$\hat{x}_{n+1} = x_n + \Delta_t \hat{v}_{n+1/2}$$

has $O(\Delta_t^3)$ error

Then we can obtain v_n with $O(\Delta_t^2)$ error: $v_n = (\hat{x}_{n+1} - x_{n-1})/(2\Delta_t)$

$v_n = (\hat{x}_{n+1} - x_{n-1})/(2\Delta_t) + O(\Delta_t^2)$

Now we can use this in the acceleration $a_n(x_n,v_n,t)$; $O(\Delta_t^2)$ error

Summary of procedure:

$$\hat{v}_{n+1/2} = v_{n-1/2} + \Delta_t[F(x_n,t_n) - G(v_{n-1/2})]/m$$

$$\hat{x}_{n+1} = x_n + \Delta_t \hat{v}_{n+1/2}$$

$$v_n = (\hat{x}_{n+1} - x_{n-1})/(2\Delta_t)$$

$$v_{n+1/2} = v_{n-1/2} + \Delta_t a_n$$

v_n used here in a_n

$$x_{n+1} = x_n + \Delta_t v_{n+1/2}$$

More than twice as much work as the standard Leapfrog method

$$v_{n+1/2} = v_{n-1/2} + \Delta_t a_n$$

$$x_{n+1} = x_n + \Delta_t v_{n+1/2}$$

but the $O(\Delta_t^4)$ error is now maintained (work pays off)

Test by running friction.ipynb on the web site (tomorrow's discussion)