Binder ratio $Q = \frac{\langle m^2 \rangle}{\langle |m| \rangle^2}$

Useful dimensionless quantity for accurately locating Tc

Infinite-size behavior:

$$\langle m^2 \rangle \sim t^{-\gamma}$$
$$\langle |m| \rangle \sim t^{-\gamma/2}$$

Implies finite-size scalings

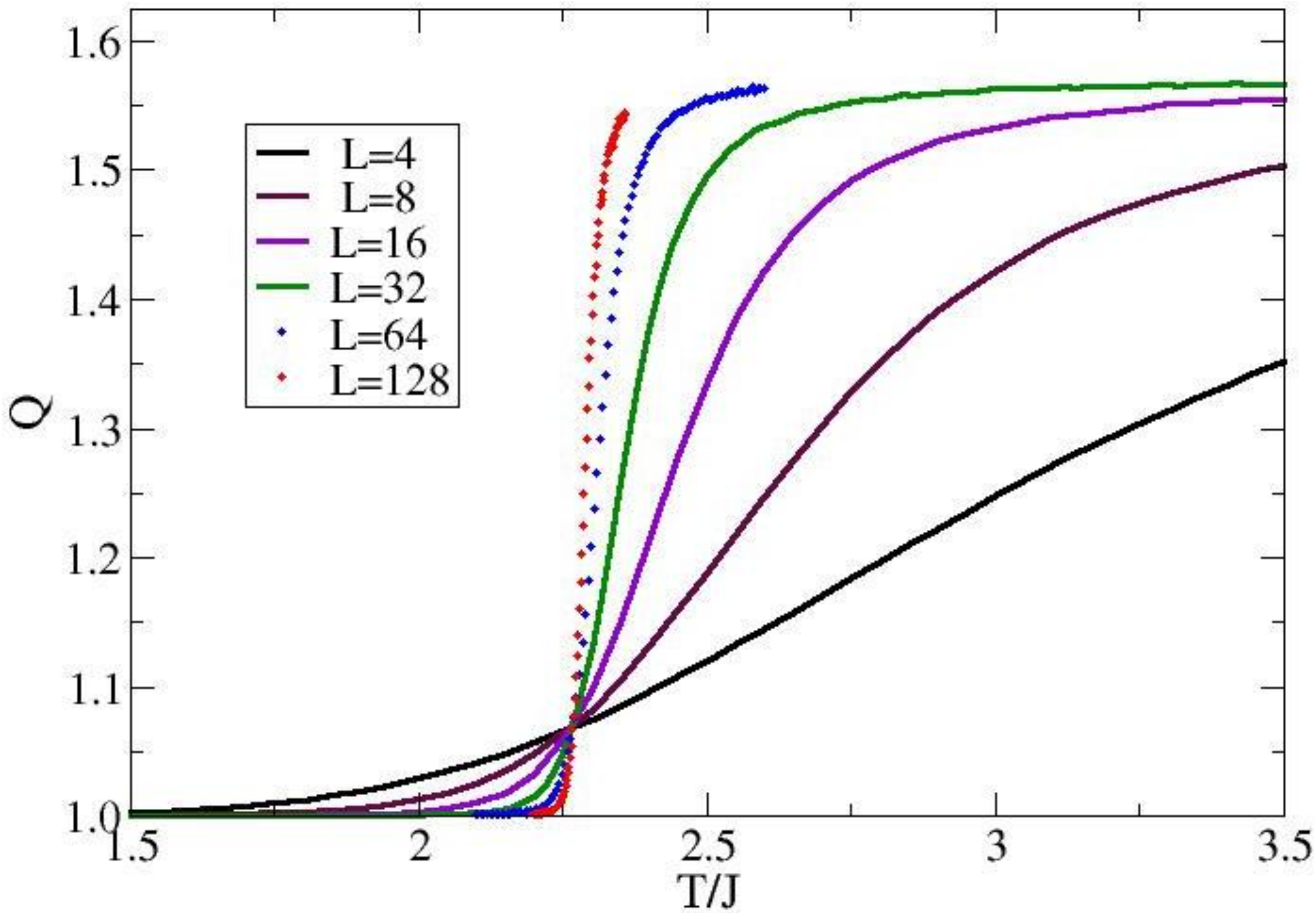
$$\langle m^2 \rangle \sim L^{\gamma/\nu}$$
$$\langle |m| \rangle \sim L^{\gamma/2\nu}$$

Hence Q should be size-independent at the critical point

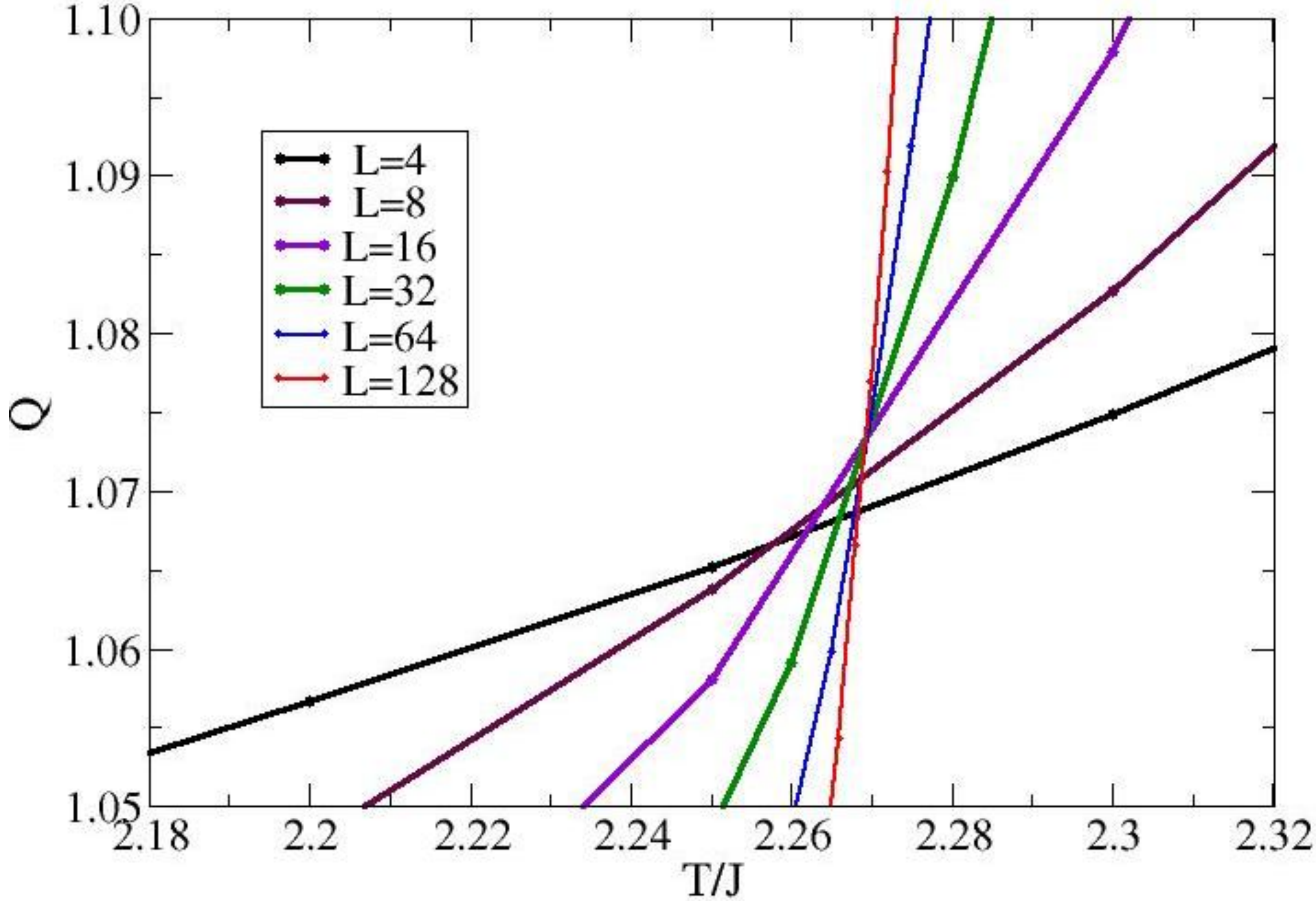
$$Q \to 1 \text{ for } T \to 0, \qquad Q \to \text{ constant for } T \to \infty$$

Q(L) curves for different L cross at Tc; often small corrections

Binder ratio: $Q = \frac{\langle m^2 \rangle}{\langle |m| \rangle^2}$



Q is size independent at Tc (useful for locating Tc)



Crossing points for, e.g., sizes L, 2L can be extrapolated to infinite L to give an accurate value for Tc

- in many cases: sufficient accuracy for two large sizes

Autocorrelation functions

Value of some quantity at Monte Carlo step i: Q_i

The autocorrelation function measures how a quantity becomes statistically independent from its value at previous steps

$$A_Q(\tau) = \frac{\langle Q_{i+\tau} Q_i \rangle - \langle Q_i \rangle^2}{\langle Q_i^2 \rangle - \langle Q_i \rangle^2}$$ (time averages)

Asymptotical decay

$$A_Q(\tau) \sim \mathrm{e}^{-\tau/\Theta}, \quad \Theta = \text{ autocorrelation time}$$

Integerated autocorrelation time

$$\Theta_{\text{int}} = \frac{1}{2} + \sum_{\tau=1}^{\infty} A_Q(\tau)$$

Critical slowing down

$$\Theta \to \infty \text{ as } T \to T_c$$

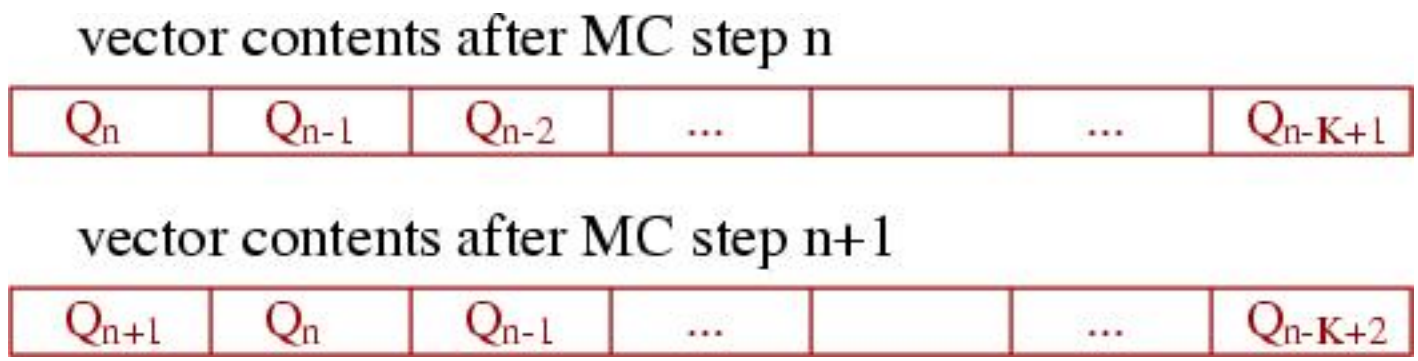
At a critical point for system of length L; Q=order parameter

$$\Theta \sim L^z, \quad z = \text{ dynamic exponent}$$

How to calculate autocorrelation functions

If we want autocorrelations for up to K MC step separations, we need to store K successive measurements of quantity Q

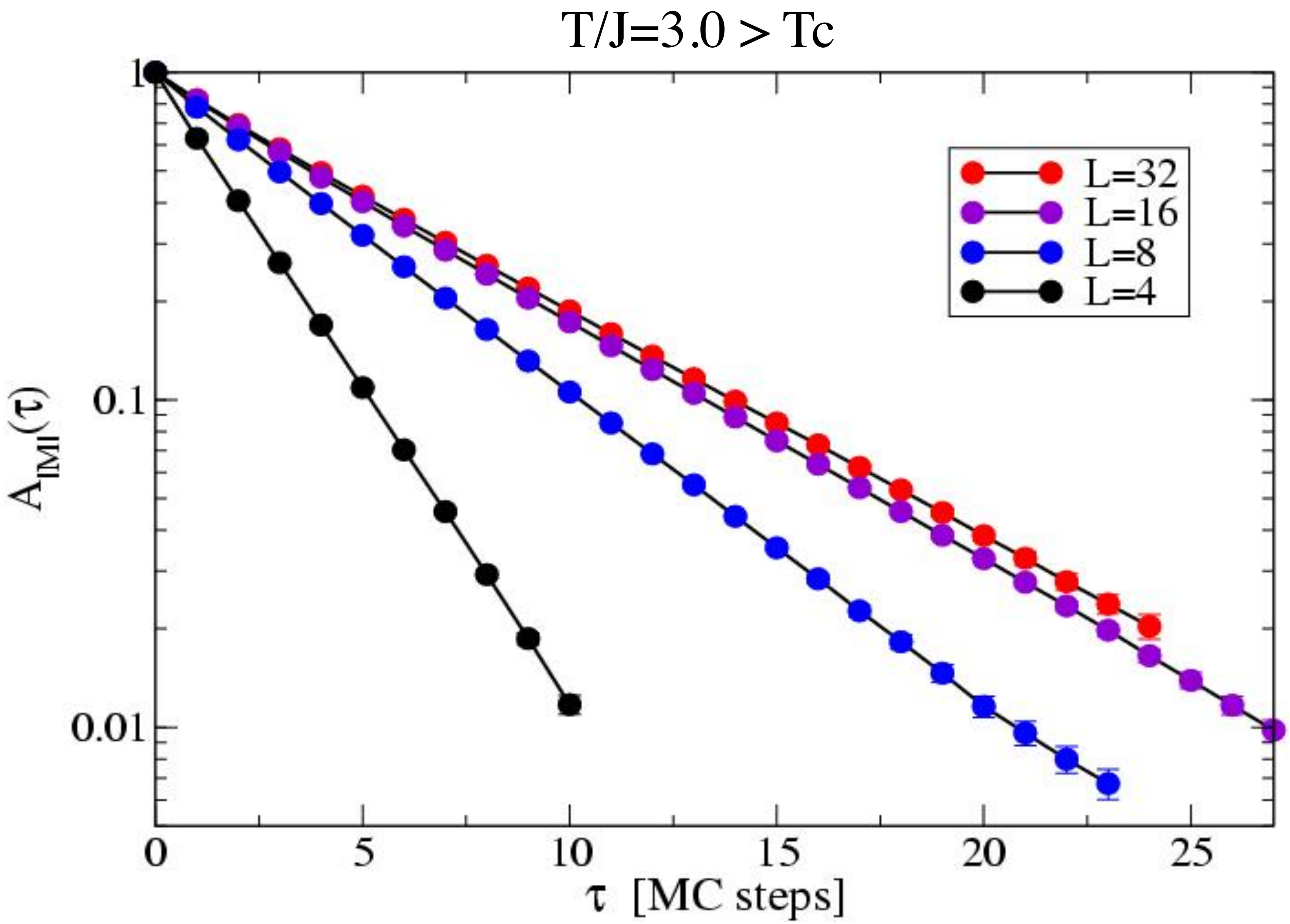
Store values in vector `tobs(K)`; first k steps to fill the vector. Then, shift values after each step, add latest measurement:



Accumulate time-averaged correlation functions

```
do t=2,k
   tobs(t)=tobs(t-1)
enddo
tobs(1)=q
do t=0,k-1
   acorr(t)=acorr(t)+tobs(1)*tobs(1+t)
enddo
```

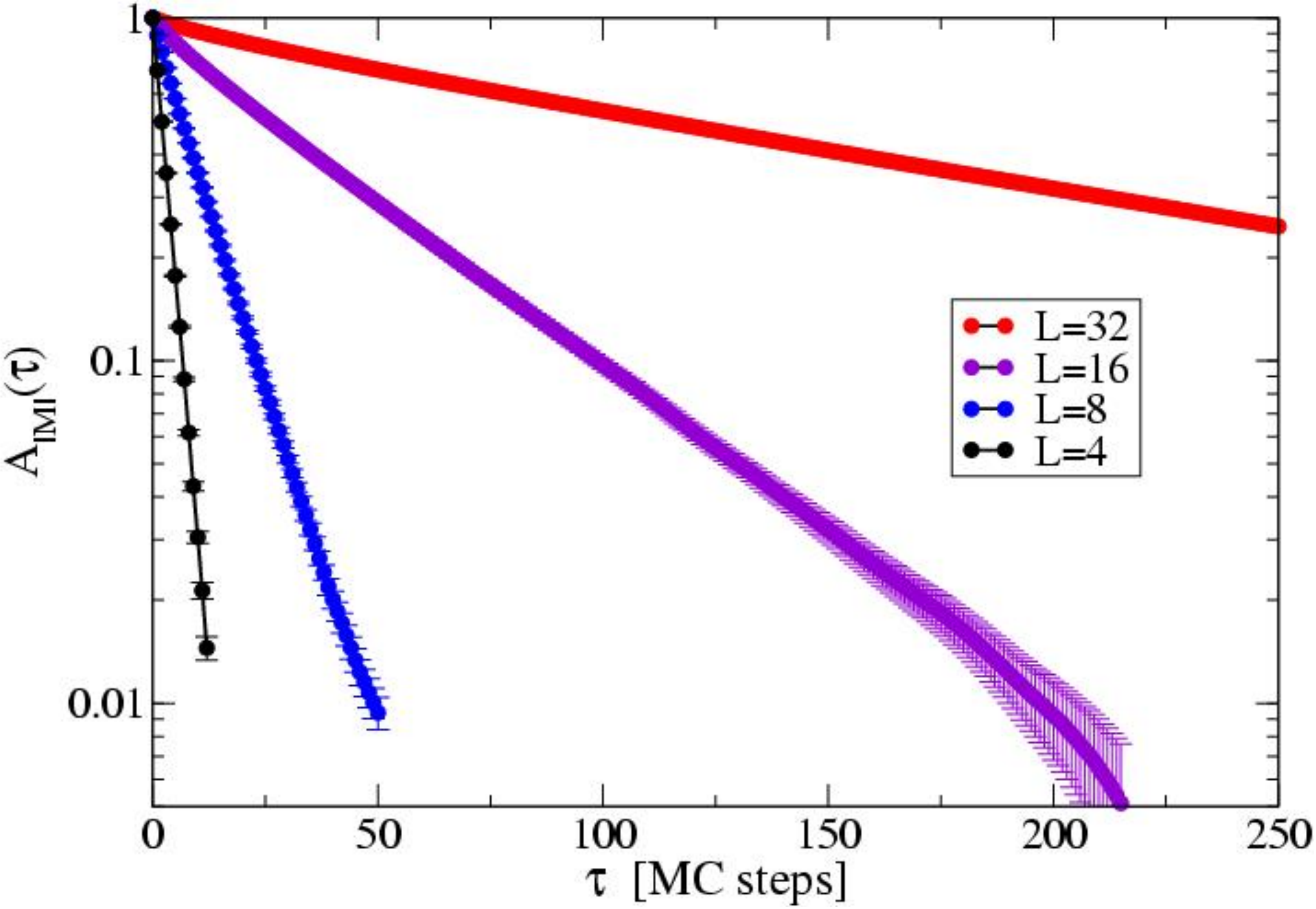
2D Ising autocorrelation functions for |M|



Exponentially decaying autocorrelation function

- convergent autocorrelation time

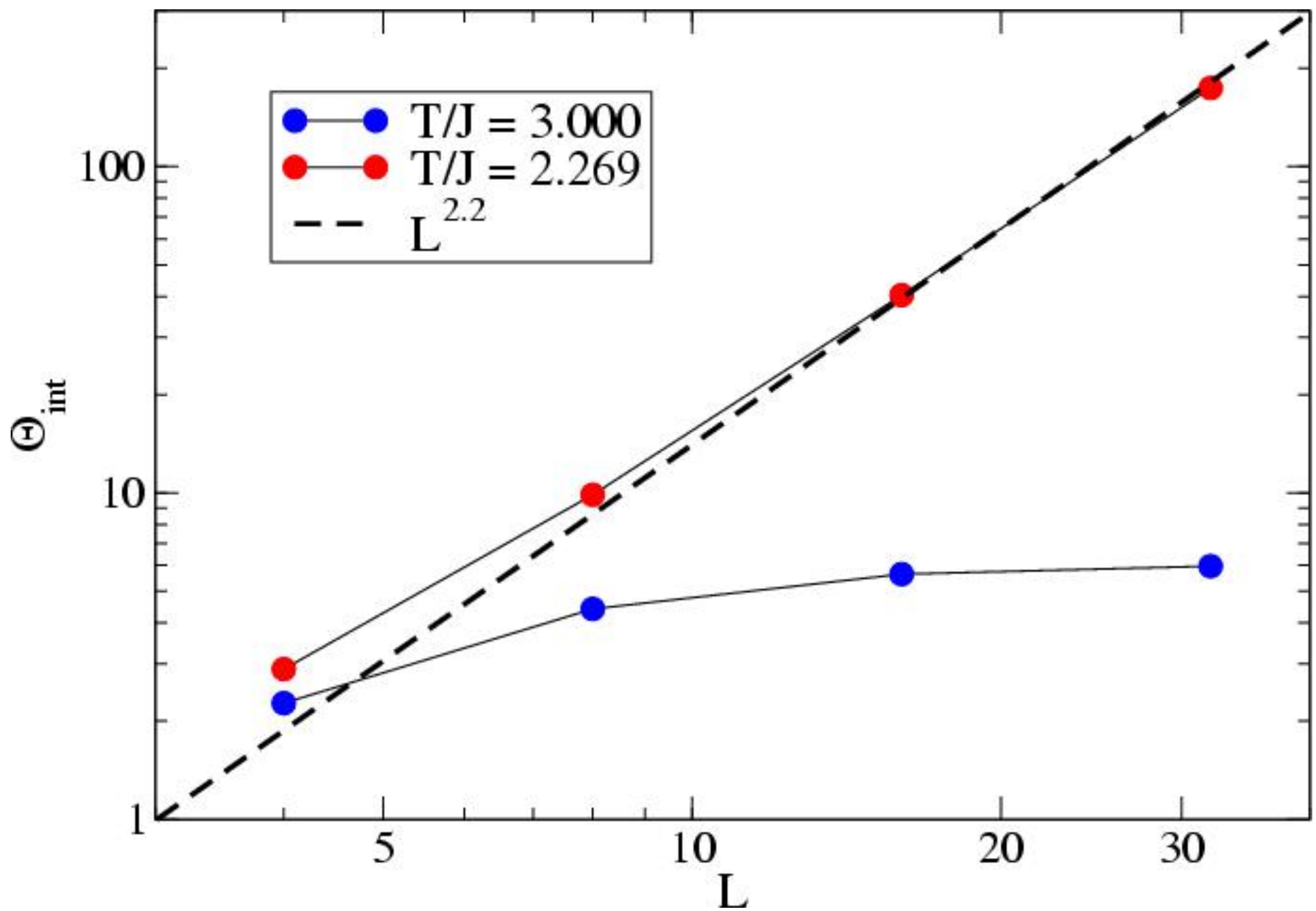
T/J = 2.269 = Tc



Autocorrelation time diverges with L

Critical slowing down

Dynamic exponent Z: $\Theta, \Theta_{\text{int}} \sim L^Z$



For the Metropolis algorithm (Glauber dynamics) $Z \approx 2.2$