Homework 3; due on Tuesday, October 15

PY 502, Computational Physics, Fall 2024

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PERTURBATION OF A GEO-STATIONARY SATELLITE ORBIT

Here you will analyze the effects of the moon on the orbit of a satellite in a geo-stationary orbit, by numerically solving the equations of motion of the satellite moving in the combined gravitational field of the earth and the moon. You are then going to find the orbit that best approximates a geo-stationary one in the presence of the moon.

In an ideal geo-stationary orbit, a satellite appears to hang motionless over a spot on the equator; the time needed for it to complete an orbit is the same as earth's rotation period (i.e., the siderial day, which is approximately 4 minutes shorter than the solar day). The properties of the geo-stationary orbit are easily found from Newton's equations for circular Kepler orbits: The downward gravitational force exerted by earth on a satellite of mass m is

$$F = \frac{GM_e m}{r^2}, \tag{1}$$

where r is the distance from earth's center to the satellite, M_e is earth's mass, $M_e = 5.9736 \cdot 10^{24}$ kg, and the constant of gravitation $G = 6.6743 \cdot 10^{-11}$ m^3/kgs^2. The centrifugal acceleration vector is directed away from earth's center and its magnitude is given by

$$a = \frac{v^2}{r}, \tag{2}$$

where v is the velocity. By equating F/m and a we can obtain the velocity in a circular orbit;

$$v = \sqrt{\frac{GM_e}{r}}, \tag{3}$$

and hence the period of the orbit is

$$T = \frac{2\pi r}{v} = \sqrt{\frac{4\pi^2 r^3}{GM_e}}. \tag{4}$$

The radius as a function of the period is thus given by

$$r = \left(\frac{GM_e T^2}{4\pi^2}\right)^{1/3}. \tag{5}$$

To obtain the geo-stationary orbit, we set the period T equal to the siderial day, for which we use the value $T_s = 86164$ s, which gives $r_{\rm geo} \approx 42168$ km.

The above simple calculation of the geo-stationary orbit neglects several effects that are important in practice. The most obvious disturbance is due to the moon. Although the moon is on average

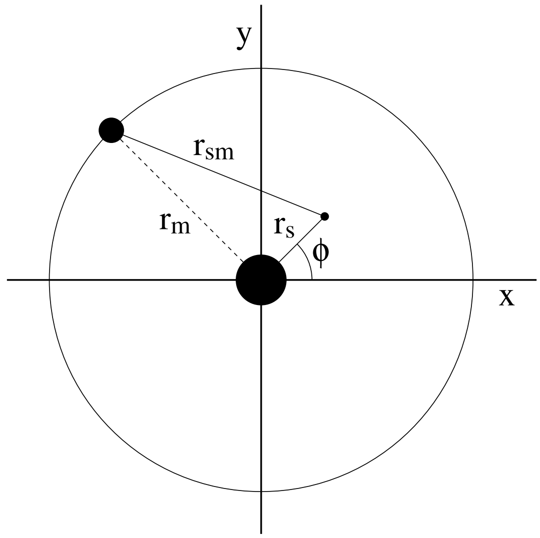

Figure 1: Schematic illustration of a satellite moving in the combined gravitational fields of the earth and the moon. In a first approximation, earth is assumed to be a stationary body around which the moon is in a circular orbit around the equator. The distances r_m, r_s, and r_{sm} are measured from the center of the moon and earth.

roughly ten times further away from the satellite than the earth, and is roughly 80 times lighter, it is clear that the satellite's orbit must periodically be corrected because of the presence of the moon (and other factors, such as the presence of some atmosphere even at very high altitudes). But how big is this effect?

We will here first, in part A) below, assume that the moon is in a perfect circular orbit around earth's equator, at a constant distance of $r_m = 384400$ km from earth's center (its actual average distance) with a period T_m equal to its actual period 27.32 solar days, or $T_m = 27.25 \cdot T_s$. The mass of the moon is $7.3477 \cdot 10^{22}$ kg, so the corresponding gravitational factor for the moon can be written as $GM_m = 1.2300 \cdot 10^{-2} \cdot GM_e$. With the moon orbiting in the equatorial plane, the motion of the satellite is also confined to this plane. This two-dimensional situation is schematically illustrated in Fig. 1. The total force experienced by the satellite is that due to the earth, at separation $\vec{r}_s$, and the moon, at separation $\vec{r}_{sm}$. The moon moves at a constant angular velocity around the earth, with the period T_m given above, and so the moon's orbit just follows a simple formula and is not solved for by the equations of motion. Earth is assumed to be a stationary object. We would like to know to what extent the presence of the moon influences the location if the satellite above earth, which is essentially quantified in polar coordinates (r, ϕ) as the difference Δ_ϕ as a function of time of the angle ϕ with and without the moon present.

In reality the lunar orbit is not in the equatorial plane (as you have undoubtedly noticed). Due to the complex mechanics of the sun-earth-moon system, the inclination of the lunar orbit (i.e., the angle beween it and Earth's equatorial plane), is not constant but fluctuates between $\approx 25^\circ \pm 5^\circ$. Here, in part B) of the assignment, we will for simplicity assume that the inclination α is constant; $\alpha = 25^\circ$. Because of the non-zero inclination, the satellite will not be confined to the equatorial plane either. The geometry of this refined three dimensional earth-moon-satellite system is described in Fig. 2. One can now ask how large the satellite's deviations from the equatorial plane are (the time-

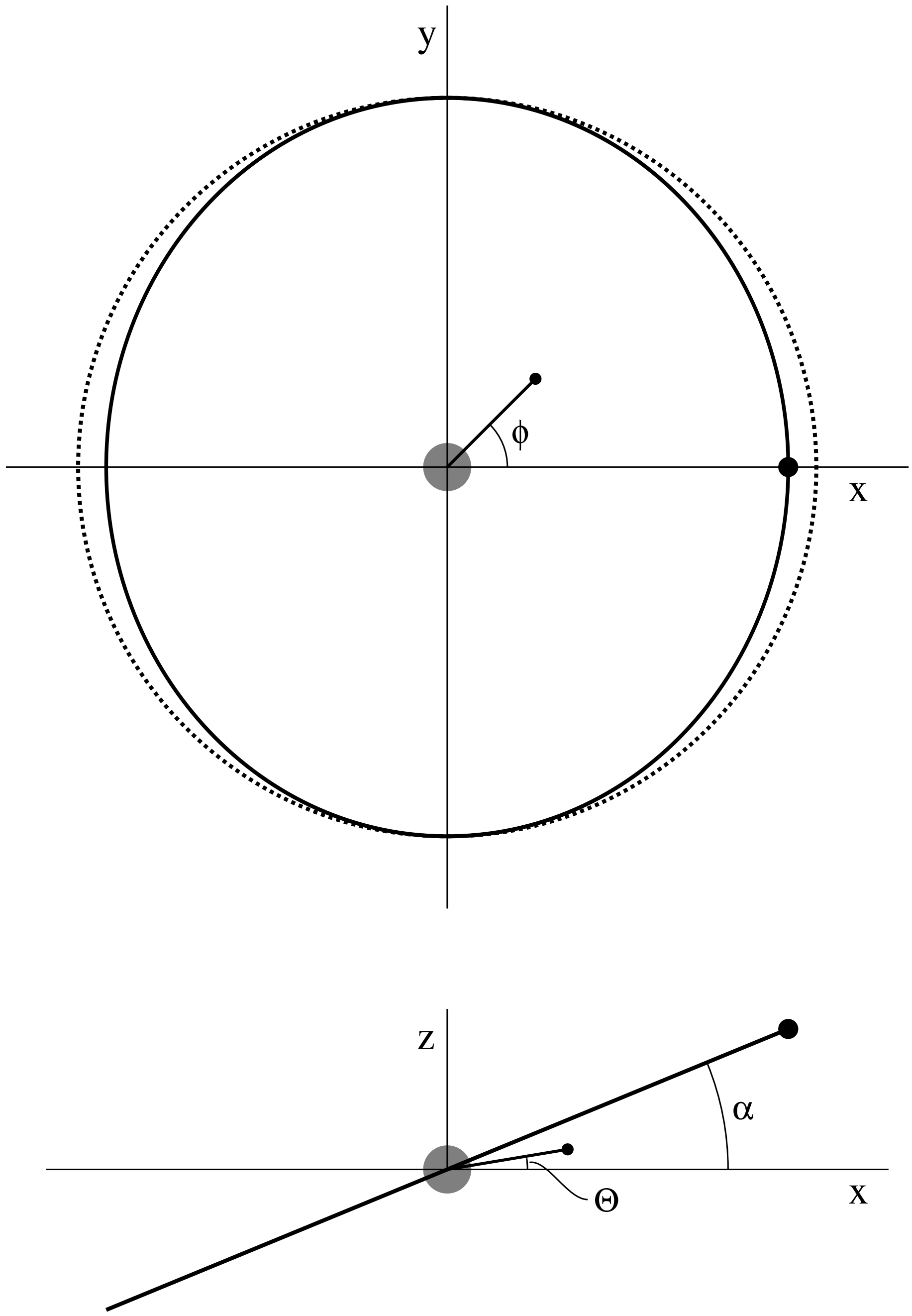

Figure 2: Geometry of the earth-moon-satellite system in the case of the moon's orbit not being in the equatorial plane. The top graph shows the projection (solid curve) of the lunar orbit onto the xy-plane (the equatorial plane). The outer dashed circle shows the orbit for zero inclination, i.e., when the moon's orbit is in the equatorial plane. The satellite's longitudinal position angle ϕ is defined with respect to the positive x-axis. The bottom graph shows a side-view of the system, defining the inclination α of the moon's orbit and the angular deviation Θ of the satellite from the equatorial plane.

dependent angle Θ in Fig. 2), and how the inclination affects the angular deviation Δ_ϕ.

Programming task

You should write a single program in which the inclination angle α is a variable to be given by the user. You can then use it to study both $\alpha = 0$ (part A) and $\alpha = 25°$ (part B). The program should use the leapfrog algorithm to integrate the equations of motion of the satellite as it experiences the combined gravitational forces due to the earth and the moon,

$$\frac{\vec{F}}{m} = -\frac{GM_e}{r_s^3}\vec{r}_s - \frac{GM_m}{r_{sm}^3}\vec{r}_{sm}, \tag{6}$$

where $\vec{r}_s$ is the position of the satellite relative to earth and $\vec{r}_{sm}$ is its separation from the moon; $\vec{r}_{sm} = \vec{r}_s - \vec{r}_m$ (as indicated in Fig. 1). The position of the moon is given explicitly by

$$\vec{r}_m = R_m \cos(\alpha)\cos(2\pi t/T_m)\vec{e}_x + R_m \sin(2\pi t/T_m)\vec{e}_y + R_m \sin(\alpha)\cos(2\pi t/T_m)\vec{e}_z, \tag{7}$$

where $T_m = 27.25 \cdot T_s$.

The initial conditions are the following: The satellite is injected at angle $\phi = 0$ at a distance from earth's center equal to the radius of a circular orbit with $T = aT_s$ (for a geo-stationary orbit $a = 1$, but you will also consider other initial conditions corresponding to $a \neq 1$). Thus the initial values of the co-ordinates are: $x_0 = r(T)$ with $r(T)$ given by Eq. (5), and $y_0 = z_0 = 0$. The initial velocity components are $v_x = v_z = 0$ and $v_y = v[r(T)]$ calculated according to Eq. (3). The position of the moon at the time of injection is $x = R_m\cos(\alpha), y = 0, z = R_m \sin(\alpha)$, which follows automatically from (7) at time $t = 0$ and is in accord with Fig. 2. Both the satellite and the moon move counter-clockwise in the figures (ϕ increasing with time).

Use the in the Cartesian (x, y, z) co-ordinate system defined in the figures to solve the equations of motion. Define the time step δ_t to be a fraction of the siderial day; $\delta_t = T_s/N_t$, with N_t a user-supplied integer. The integration time $t_{\max}$, which we also take to be an integer multiple of T_s, should also be supplied by the user.

The program should output the position of the satellite as a function of time in polar coordinates (use a counter n_r for the total number of revolutions around earth, i.e., the number of time the angle has crossed over $\phi = 2\pi$, and output the accumulated angle $2\pi n_r + \phi$). The deviations from the geo-stationary orbit $[\phi_0(t), r_{\text{geo}}]$ should also be calculated. The angular difference is given by

$$\Delta_\phi = \phi(t) - \phi_0(t) = \phi(t) - 2\pi\frac{t}{T_s}, \tag{8}$$

and the radial difference is

$$\Delta_r(t) = r(t) - r_{\text{geo}} = r(t) - \left(\frac{GM_e T_s^2}{4\pi^2}\right)^{1/3}. \tag{9}$$

Also calculate the angle Θ in Fig. 2, which characterizes the deviation from the equatorial plane in the case of the non-zero inclination of the moon. Convert $r(t)$ and $\Delta_r(t)$ to kilometers for the output. The output should be written to a file `sat.dat` in the form of lines for each t containing

$$t,\ \phi(t),\ r(t),\ \Delta_\phi(t),\ \Delta_r(t), \Theta(t). \tag{10}$$

Since the time step δ_t used in the leapfrog algorithm may have to be much smaller than what we are interested in when making a graph, you should only write data every N_w steps, where N_w is to be supplied by the user. All input data should be read from the keyboard.

Specific issues to investigate

A) Effects of the "equatorial moon" on a geo-stationary orbit

In the absence of the moon, a satellite injected into an orbit with $T = T_s$ would move for ever in the geo-stationary orbit, and its angular position would simply be $\phi_0(t)$ as discussed above. Run your program with $T = T_s$ for a total time $t_{\text{int}} = 100$ days. Produce a graph showing the deviations Δ_ϕ and Δ_r as functions of t. Can the effects of the moon be neglected when operating the satellite?

You should also provide evidence that your integration of the equations of motion is done with a sufficuently small time-step δ_t. In addition to the graph with your final result, also produce a graph of results for the time window 90-100 days obtained using different δ_t. Roughly what δ_t do you judge to be sufficiently small?

The deviations from the geo-stationary orbit can be made smaller by injecting the satellite with slightly different initial conditions, corresponding to $T = aT_s$ with $a \neq 1$. Adjust a so that you minimize the over-all drift in Δ_ϕ over a 500 day time span, i.e., find the initial conditions that give the best realization of an almost geo-stationary orbit when the effects of the moon are taken into account. You can do this by trial and error, i.e., running the program several times for different values of a and adjusting/bracketing it until you have minimized the deviations to an accuracy corresponding approximately to what can be resolved in a graph. Produce graphs showing $\Delta_\phi(t)$ up to $t = 50$ days and up to 500 days (two different plots just to make features on both large and small time-scales apparent). Indicate the best value $a = T/T_s$ that you found and the corresponding initial radius r/r_{geo}. Can you explain, semi-quantitatively, the physical reason for the optimum a?

B) Effects of the "25° inclined moon" on a geo-stationary orbit

Now repeat the simulations with inclination $\alpha = 25°$, for which the motion becomes three-dimensional. Produce a graph showing the deviation Θ of the satellite from the equatorial plane and the deviation Δ_ϕ over a 200 day period. Then, in a way similar to Problem A), adjust the periodicity T of the initial orbit to find the injection parameters r, v that produce the best geo-stationary orbit over a 500 day period [state the best r and produce plots of $\Delta_\phi(t)$ and $\Theta(t)$].