Momentum Space RG I

Thus far in the class we have emphasized real space methods, now we will turn to the type of methods that are more widely actually used in practice

- momentum space RG (Feynman diagrams, QFT techniques)
- Numerical methods (NRG, DMRG, etc...)

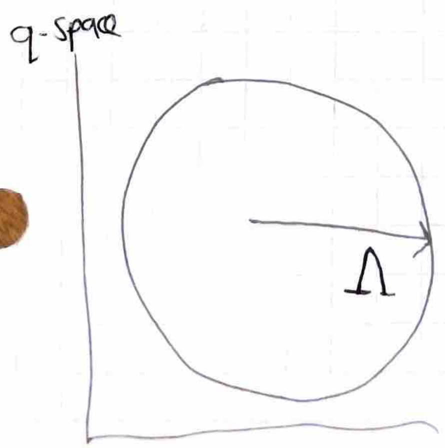
Let us start by thinking about the ϵ-expansion in momentum space to think about how we can think about RG in $d<4$ in Ising model...

Recall RG procedure consists of two steps

1. Decimation
2. Rescaling

So how do we do this in momentum space...

We will think about this in terms of "long wavelength" cutoff of momentum Λ (which we can think of as $\Lambda \sim \frac{2\pi}{a}$ ← lattice spacing)

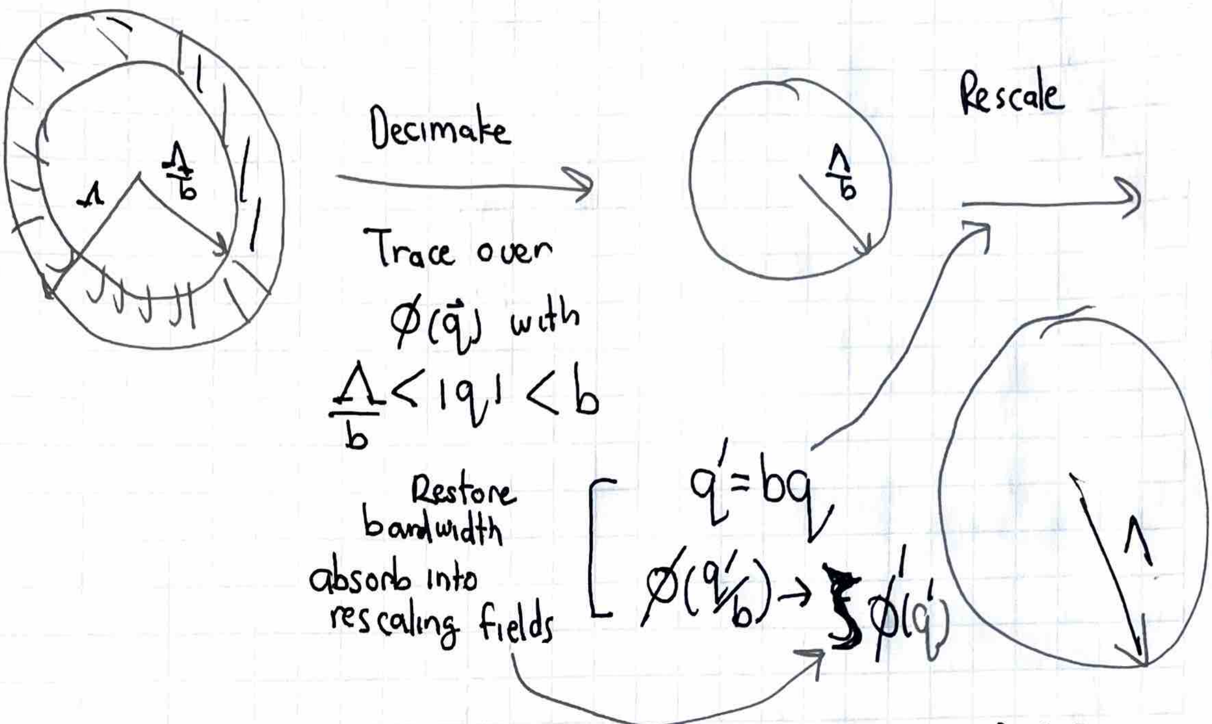


We will then reduce the cutoff

$$\Lambda \longrightarrow \frac{\Lambda}{b}$$

by "integrating out" long wavelength fluctuations

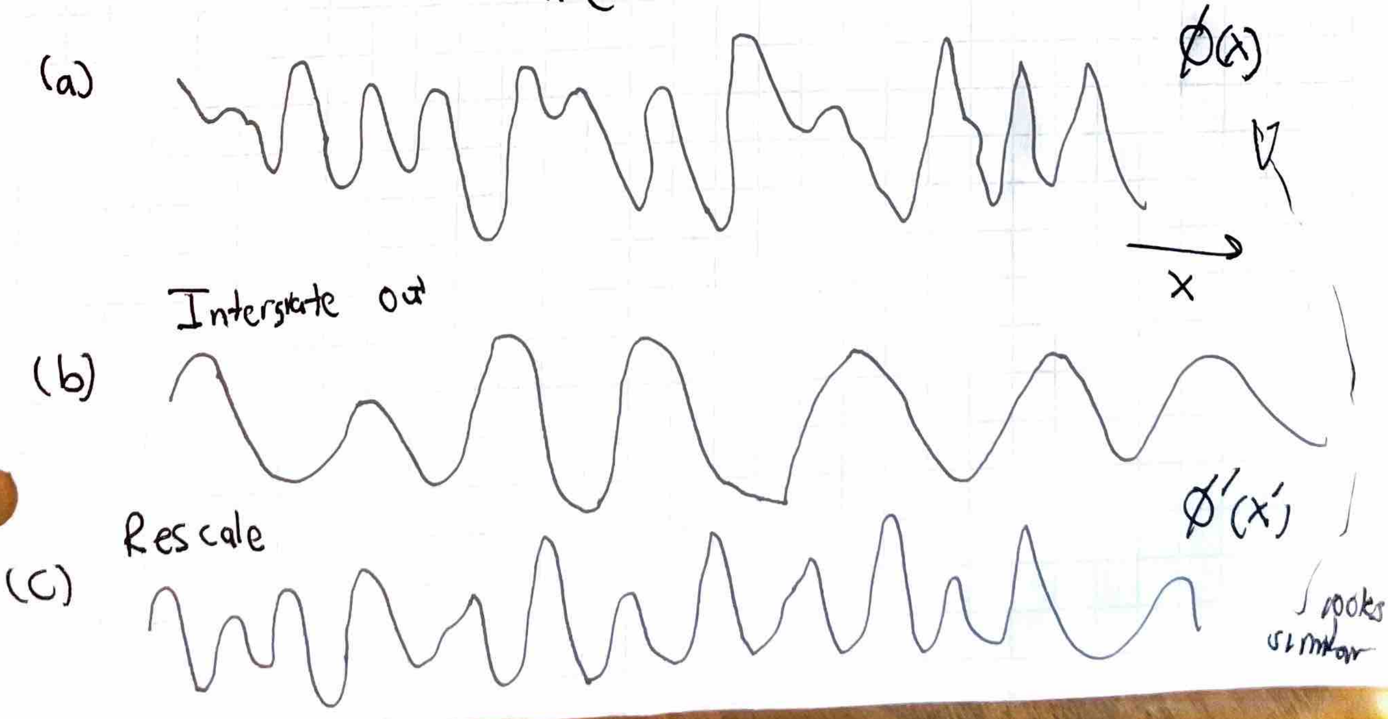
The RG procedure essentially looks like



This is just the same procedure we did in Real Space when we calculated "naive" scaling dimensions ... Recall that

ζ can acquire "anamolous dimension"

What does this look like



To do this we will just divide into "high" and "low" wave number components

$$\phi(q) = \phi^<(\vec{q}) + \phi^>(\vec{q})$$

$$\phi^<(\vec{q}) = \begin{cases} \phi(\vec{q}) & \text{if } 0 < q < \frac{\Lambda}{b} \\ 0 & \text{if } \frac{\Lambda}{b} < q < \Lambda \end{cases}$$

$$\phi^>(\vec{q}) = \begin{cases} 0 & \text{if } 0 < q < \frac{\Lambda}{b} \\ \phi(\vec{q}) & \text{if } \frac{\Lambda}{b} < q < \Lambda \end{cases}$$

Notice $\phi(\vec{q}) = \phi^<(\vec{q}) + \phi^>(\vec{q})$

So decimation is just

$$e^{-H_{\Lambda/b}[\phi^<(q)]} = \int \mathcal{D}\phi^>(\vec{q})\, e^{-\bar{H}_\Lambda[\phi^<(\vec{q}) + \phi^>(\vec{q})]}$$

$\int_0^{\Lambda/b} d^dq \rightarrow$ rescale $q' = bq \Longrightarrow b^{-d} \int_0^{\Lambda} d^dq'$

Then $\}\ \phi'(q') = \phi^<(q'/b)$

$\uparrow$ usual volume factor..

Together this defines RG transformation..

$$\bar{H}_\Lambda[\phi'(q')] = R_b H_\Lambda[\phi(q)]$$

How can we use this to actually calculate stuff?

Let us start with scaling of the correlation functions

$$\langle \phi(q_1)\phi(q_2)\rangle_H = \frac{1}{Z}\int D\phi(q)\, e^{-\overline{H}_\Lambda[\phi(q)]}\phi(q_1)\phi(q_2)$$

$Z = \int D\phi(q) e^{-\overline{H}_\Lambda[\phi(q)]}$ (original Hamiltonian)

$$= \frac{1}{Z'}\int D\phi'(q')\, e^{-\overline{H}_\Lambda'[\phi'(q')]}\, \zeta^2 \phi'(bq_1)\phi'(bq_2)$$

$$= \zeta^2 \langle \phi'(bq_1)\phi'(bq_2)\rangle_{H'}$$

For translationally invariant system

$$\langle \phi(\vec{q}_1)\phi(\vec{q}_2)\rangle = G(q)\, 2\pi^d\, \delta(\vec{q}_1+\vec{q}_2) \quad \vec{q} = \vec{q}_1 - \vec{q}_2$$

$$\langle \phi'(bq_1)\phi'(bq_2)\rangle_{H'} = G'(bq)\, 2\pi^d\, \delta(b\vec{q}_1 + b\vec{q}_2)$$

$$b^{-d}\delta(\vec{q}_1+\vec{q}_2)$$

$\Longrightarrow$

$$G(\vec{q}) = \zeta^2 b^{-d} G'(b\vec{q})$$

At critical point $G(q) \sim \dfrac{1}{q^{2-\eta}}$

$\Longrightarrow$

$$\frac{A}{q^{2-\eta}} = \zeta^2 b^{-d}\frac{A'}{(bq)^{2-\eta}} = \zeta^2 b^{-(d+2-\eta)}\frac{A'}{q^{2-\eta}}$$

$$\zeta^2 = b^{(d+2-\eta)}\left(\frac{A}{A'}\right)$$

However, let us choose RG scheme so $A = A'$

In this case

$$\zeta = b^{\frac{(d+2-\eta)}{2}}$$

and $G(q) = A q^{-(2-\eta)}$ at fixed point... This is just same scaling we already discussed in real space...

Gaussian Model

Let us re-derive the scalings for the Gaussian model

$$\bar{H}_{0,\Lambda} = \frac{1}{2}\int_0^{\Lambda} \frac{d^dq}{(2\pi)^d}(r + cq^2)\,|\phi(\vec{q})|^2 + \underbrace{V_0}_{\text{constant}}$$

$$= \frac{1}{2}\underbrace{\int_0^{\Lambda/b} \frac{d^dq}{(2\pi)^2}}_{\int_q^<}(r + cq^2)\,|\phi^<(q)|^2$$

$$+ \frac{1}{2}\underbrace{\int_{\Lambda/b}^{\Lambda} \frac{d^dq}{(2\pi)^2}}_{\int_q^>}(r + cq^2)\,|\phi^>(q)|^2 + V_0$$

So now

$$e^{-\bar{H}_{\Lambda/b}[\phi^<(\vec{q})]} = e^{-\bar{H}^<_{0,\Lambda}} \int D\phi^>(\vec{q})\, e^{-H_{0,\Lambda}[\phi^>(\vec{q})] + V_0} e$$

$$\equiv e^{-\bar{H}^<_{0,\Lambda} + \underbrace{V_0 - \ln Z_0^>}_{F_b}}$$

$$\equiv e^{-F_b}\, e^{-\bar{H}^<_{\Lambda}[\phi^<(q)]}$$

✓ ~~(Replaced integral w/ discrete sum?)~~

Thus, we have

$$Z = e^{-F_>} \int \mathcal{D}\phi^< \, e^{-\left[-\frac{1}{2} \int^{\Lambda/b} \frac{d^dq}{(2\pi)^d} (q^2 + r)\, \phi(\vec{q})\, \phi(-q)\right]}$$

So now $\Lambda' = b\Lambda \Rightarrow \phi'_q = b^{\frac{(d-2)}{2}} \phi_q$

To see this note

$$H^<[\phi^<(q)] \int_0^{\Lambda/b} \frac{d^dq}{(2\pi)^d} (q^2+r)\, \phi^<(-q)\, \phi^<(q)$$

$$q' = bq$$

$$\sim \int \frac{d^dq'}{(2\pi)^d} b^{-d+2} q^2\, \phi^<\!\left(\frac{-q}{b}\right) \phi\!\left(\frac{q'}{b}\right) + \int \frac{d^dq}{(2\pi)^d} b^{-d} r_0\, \phi^<\!\left(\frac{-q'}{b}\right) \phi^<\!\left(\frac{q'}{b}\right)$$

$$\sim \int \frac{d^dq'}{(2\pi)^d} \left[b^{-d+2} \xi^2 q'^2 \phi(q')\phi(q') + b^{-d} \xi^2 r_0\, \phi(q')\phi(-q') \right]$$

Free energy $\sim$ Invariant

$$\xi \sim b^{\frac{-2-d}{2}} \Rightarrow \underbrace{r_0 b^2}_{r'}$$

$$r' = b^2 r$$

This implies

$$\xi(r) = b\,\xi(r') = b\,\xi(b^2 r)$$

$$\Rightarrow \xi \sim \frac{1}{r^{1/2}} \sim \frac{1}{|T-T_c|^{\frac{1}{2}}} \Rightarrow \boxed{\nu = \frac{1}{2}}$$

We will often be interested in case

$$b = (1 + \delta \ell)$$ "Infinitesimal"

$$\frac{dr(\ell)}{d\ell} = \beta(r)$$ "Beta function"

$$= 2r(\ell)$$

$$\Longrightarrow \quad r(\ell) = e^{2\ell} r(0)$$

$$e^{\ell} = \left[\frac{r(\ell)}{r(0)}\right]^{1/2}$$

This is non-trivial flow equation