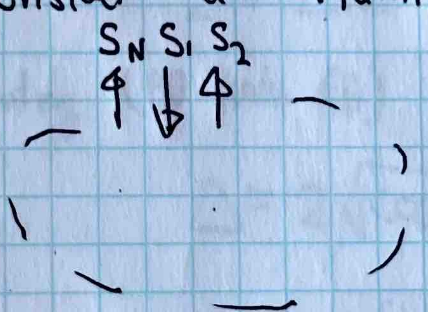
Real Space RG For Ising Model

Let us return to the Ising model and do some real space RG...

There is only one system we can really do RG exactly, this is the 1-D Ising model. We will start by doing RG on this system and use these results to do "approximate RG" using a variational scheme in 2 dimensions...

1-D Ising Model RG

We consider a Hamitonian with periodic boundary conditions

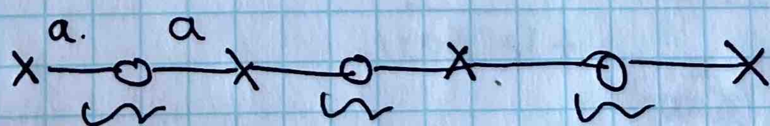


$$H = -J \sum_i S_i S_{i+1} + h \sum_i S_i$$

$$S_i = \{\pm 1\}$$

Let us start with $h=0$ and derive some RG equations by decimation..

Recall, the goal is to "rescale" the system by some scale b.. So what we will do is choose $b=2$ by



Tracing over even spins..

Notice we have changed the effective lattice spacing $a' = 2a$..

With this we can look at say the terms involving S_2 and

$$\mathrm{Tr}_{S_2} e^{\beta J S_2 (S_1 + S_3)} = 2\cosh \beta J (S_1 + S_3)$$
$$= \Delta e^{\beta J' S_1 S_3}$$

To see relations first plug in $S_1 = 1$ and $S_3 = 1$

$$2\cosh 2\beta J = \Delta e^{\beta J'} \qquad (A)$$

Now plug in $S_1 = 1$ and $S_3 = -1$

$$2 = \Delta e^{-\beta J'} \qquad (B)$$

Subtituting (B) into (A) gives

$$\Delta = 2\sqrt{\cosh 2\beta J}$$

Then we use the fact that

$$e^{\beta J'} = \sqrt{\cosh 2\beta J}$$

to show that

$$\tanh \beta J' = \frac{\sqrt{\cosh 2\beta J} - \frac{1}{\sqrt{\cosh 2\beta J}}}{\sqrt{\cosh 2\beta J} + \frac{1}{\sqrt{\cosh 2\beta J}}} = \frac{\cosh 2\beta J - 1}{\cosh 2\beta J + 1} =$$

$$= \frac{\cosh^2 \beta J + \sinh^2 \beta J - (\cosh^2 \beta J - \sinh^2 \beta J)}{\cosh^2 \beta J + \sin^2 \beta J + (\cosh^2 \beta J - \sin^2 \beta J)}$$

$$= \frac{2\sinh^2 \beta J}{2\cosh^2 \beta J} = \tanh^2 \beta J$$

This allows us to define "dual coupling" $v = \tanh \beta J$
in terms of which RG equations become

$$v' = v^2$$

with $0 < v < 1$ . . .

So there are only two fixed points

$$v^* = 1 \quad \text{or} \quad v^* = 0$$

(Ferromagnetic Fixed point) ← This clearly unstable
$J \to \infty$

So there is no phase transition in 1-D . . .

We can look at the correlation length . . .
we know

$$\xi(v') = \frac{1}{2}\xi(v)$$

$$2\xi(v^2) = \xi(v)$$

This can only happen if

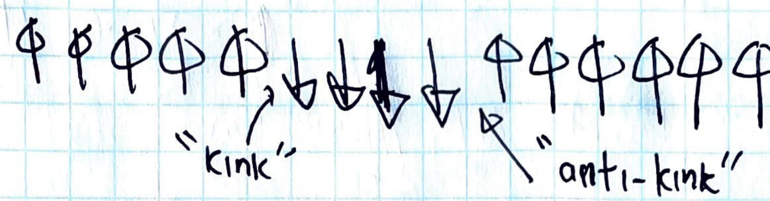
$$\xi(v) = -\frac{K}{\ln v}$$

as

$$\lim_{T\to 0} \xi(T) = \frac{-K}{\ln \tan \beta J} \Rightarrow \frac{-K}{\ln(1-2e^{-2\beta J})}$$

$$\simeq \frac{-K}{-2e^{-2\beta J}} \simeq \frac{K}{2}e^{2\beta J}$$

This exactly the Boltzman factor associated with "Kink" excitations



In the dilute limit we must balance entroy and energy

Do this for HW

You can easily show that on average

$$\xi^{-1} = \left\langle \frac{n}{N} \right\rangle^{-1} \approx e^{2J/T}$$

Destruction of long-range order by "topological" excitations... is very generic...

We can also calculate correlation length directly... to do this we note that

$$e^{\beta J S_i S_j} = \cosh \beta J + S_i S_j \sinh \beta J$$

$$= \cosh \beta J (1 + S_i S_j v) \qquad v = \tanh \beta J$$

Return to decimation calculation

$$\mathrm{Tr}_{S_2} (1 + v S_1 S_2)(1 + v S_2 S_3) = 2(1 + v^2 S_1 S_3)$$

So $G(r) = \langle S_{i+r} S_r \rangle = v^r \Rightarrow \xi^{-1} = -\ln v$

We can easily extend this argument to include magnetic field ..

$$\mathrm{Tr}_{S_2}\, e^{\beta J S_2(S_1+S_3)+\beta h S_2} = \Delta e^{\beta J' S_1 S_3 + \beta h''(S_1+S_3)}$$

Can show

(part of renormalized field)

$$\tilde{h} = \frac{1}{4\beta} \ln\left[\frac{\cosh \beta(2J+h)}{\cosh \beta(2J-h)}\right]$$

$$h' = h + 2\tilde{h}$$

$$\Delta^2 = 4\cosh\beta\left[\cosh\beta(2J+h)\cosh\beta(2J-h)\right]^{\frac{1}{2}}$$

$$J' = \frac{1}{4\beta}\ln\left[\frac{\cosh\beta(2J+h)\cosh\beta(2J-h)}{\cosh^2\beta h}\right]$$

For small h, can approximate by expanding to first order in h and get

$$\tanh\beta J' = (\tanh\beta J)^2 \Rightarrow v' = v^2$$

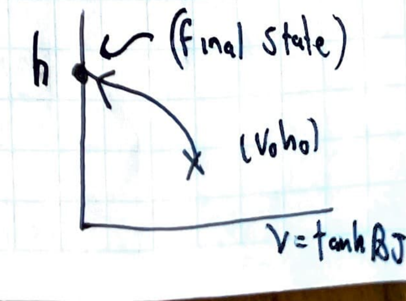
$$h' = h(1+\tanh 2\beta J)$$

Fixed points are $J^* = 0$ and h^* arbitrary and $J^* = \infty$ and $h = 0$

If one also writes $w = \tanh\beta h$ one can also write Full RG equations in a simpler form

$$\frac{1+w'}{1-w'} = \frac{1+w}{1-w}\,\frac{1+2wv+v^2}{1-2wv+v^2}$$

$$(1+v')^2 = \frac{(1+v^2)^2 - 4v^2w^2}{1+v^2}$$



At the <u>paramagnetic fixed point</u> $J^* = 0 \Rightarrow v^* = 0$

$$w' = w$$

$$v' = v^2(1 - w^2)$$

So let us calculate scaling dimension is usual way

$$y_T = \frac{\ln\left[\frac{\partial v'}{\partial v}\right]}{\ln 2} = -\infty \quad \text{(irrelevant)}$$

everything flow to paramagnet

$$y_h = \frac{\ln\left[\frac{\partial w'}{\partial w}\right]}{\ln 2} = 0$$

Field is irrelevant

At the <u>ferromagnetic fixed point</u> $v^* = 1$ and $w^* = 0$

$$1 + 2w' = (1 + 2w)(1 + 2w) \quad (\text{expand } v^* = 1 \ w^* = 0)$$

$$w' = 2w \qquad \boxed{w' = bw}$$

And writing $v = 1 - t$

$$t' = 2t \qquad \boxed{t' = bt}$$

So that $y_T = 1$ and $y_h = 1$ ...

We are now in a position to extend these trivial results to 2 dimensions....

RG of 2D Ising Model

We will now discuss an approximate RG scheme in two dimensions commonly called the Migdal-Kadanoff transformation...

This will be an <u>approximate RG</u> scheme that has some variational bounds... (Still uncontrolled but at least it has some minimum guarantees)

Like to approximate H by H' something we

$$H' = H + V$$

We have

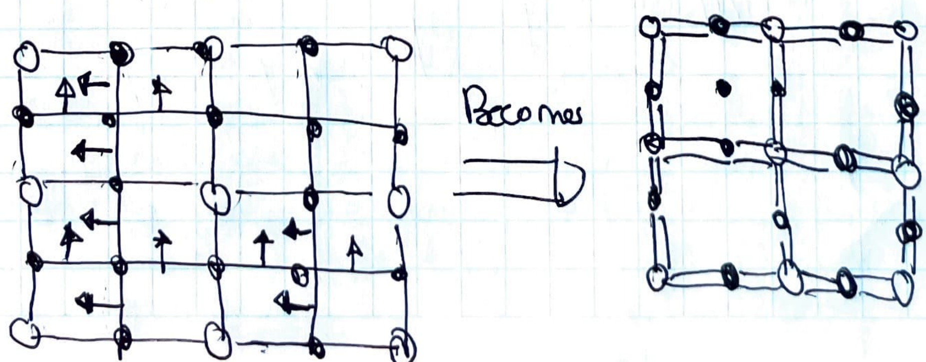
$$e^{\beta F'} = e^{-\beta F} \langle e^{\beta V} \rangle$$

$$\Rightarrow \quad e^{-\beta F'} \geq e^{-\beta F} e^{-\beta \langle V \rangle} \qquad \text{(Jensen)}$$

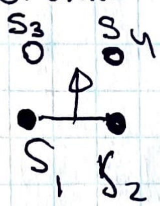
When $\langle V \rangle$ vanished $\Rightarrow$ say due to translational invariance we have that

$$F' \leq F \qquad \text{(Variation free energy is lower bound)}$$

So one can think of a decimation procedure in terms of "bond moving"



Consider moving bond between site S_1 and S_2 to S_3 and S_4



$$H = -\sum J S_1 S_2 + \delta V$$

This is the same as adding

$$\delta V = J(S_1S_2 - S_3S_4)$$

For RG scheme where $b=2$

So now we need to Trace over solid circles

This produces a new coupling between nearest neighbor spins

$$\tanh J' = \tanh^2 2J$$

(observed β into definition)

$$J' = \tanh^{-1}(\tanh^2 2J)$$

This has a fixed point $J^* \simeq 0.305$ and

$$y_T = \frac{\ln\left(\frac{\partial J'}{\partial J}\Big|_{J^*}\right)}{\ln 2} \simeq 0.748$$

(Compare with $\nu = 1$ in $d=2$)

More generally in d-dimensions

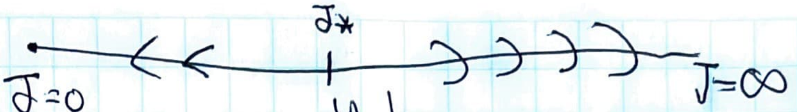
$$\tanh J' = [\tanh^b(b^{d-1}J)]$$

Now trick $b = 1 + db$

$$J' \approx J + [(d-1)J + \sinh J \cosh J \ln \tanh J]\, db$$

$$0 = \frac{dJ'}{db} = (d-1)J + \sinh J \cosh J \ln \tan J$$

Fixed point $J = J^* + J$ $\quad \frac{dJ}{db} = [d + \cosh 2J^* \ln \tanh J^*]J$



Non-trivial fixed point

$$J = J(0)\, e^{y_T b}$$

$$y_T = d + \cosh 2J^* \ln \tanh J^*$$

$d = 2$ $\quad y_T = 1.119$ and $J^* = 0.4407$

Turns out to be exact!!

Can repeat same calculation in presence of magnetic field (see Creswick 5.4) and find

$$y_h = 2d(d-1)J^*$$

$\Rightarrow \quad y_H = 1.763$ compared to 1.875 exact!

Does incredibly well...