Scaling, Hyperscaling, and RG

We can make a few more interesting arguments just from dimensional analysis...

Consider the total free energy (normalized by K_BT as usual)

We will be a little sloppy about when we absorb K_BT into couplings or not. Hopefully, clear from context!

$$\frac{F}{K_BT} = \frac{Vf}{K_BT}$$ We know that from dimensional analysis that we can write and general considerations

$$\frac{f}{K_BT} = \frac{f^{\text{non-singular}}}{K_BT} + \underbrace{\frac{f^{\text{singular}}}{K_BT}}$$

Focous on this. It has dimensions of L^{-d}

$$\left[\frac{f}{K_BT}\right] \sim L^{-d}$$

The only length scale in problem as $T \to T_c$ that diverges is $\xi(T)$ (correlation length)... This is "long-distance" physics.. Still possible to have "microscopic" scales $\ell_1, \ell_2, \ldots$

So we know that

$$\frac{f_s}{K_BT} \sim \xi^{-d}\left(A + B_1\underbrace{\left(\frac{\ell_1}{\xi_1}\right)^{\sigma_1}} + B_2\underbrace{\left(\frac{\ell_2}{\xi_2}\right)^{\sigma_2}}_{\text{dimensionless}} + \ldots\right)$$

This implies using $t = \frac{T-T_c}{T_c}$

$$\xi \sim t^{-\nu}$$

that

$$\frac{f_s}{k_B T} \sim t^{\nu d}$$

But we know that si

$$C_V = -T\frac{\partial^2 f_s}{\partial T^2} \sim (t+T_c)\frac{\partial^2 f_s}{\partial t^2} \sim t^{\nu d-2}$$

leading power

But by defintion

$$C_V \sim t^{-\alpha}$$

$$\Rightarrow \boxed{2-\alpha = d\nu}$$

(Josephson Relation)
called "hyper scaling" relation
since dimension d ...

Lets check:

Landau: $d=4$ $\quad \nu = \frac{1}{2}$ $\quad d=4$ $\quad \alpha = 0^+$

Exact

$d=2$ $\quad \alpha = 0^+$ $\quad d=2$ $\quad \nu = 1$

$d=3$ $\quad \alpha = 0.110$ $\quad d=4$ $\quad \nu = 0.630$

We can also look at the correlation function.
For simplicity assume a single microscopic length scale ℓ

$$\hat{G}(k) = \ell^{-2}\left[\frac{\xi}{\ell}\right]^{a}\left(\underbrace{\hat{g}(k\xi)}_{}\left[1 + A\left(\frac{\ell}{\xi}\right)^{\sigma} + \cdots\right]\right)$$

assumption that this is non-trivial
k-dependence

We know

$$\hat{G} \sim k^{-2+\eta} \quad \text{at } \underline{\text{critical point}} \quad T = T_c \, ..$$

Since $\xi \sim t^{-\nu}$ <u>diverges</u>, the factors of ξ
must cancel for large ξ

$$\text{So} \Rightarrow \hat{g}(k\xi) \sim \frac{1}{(k\xi)^{a}} \quad \text{and} \quad \xi \to \infty$$

and

$$\hat{G}(k) \sim \frac{\ell^{-2-a}}{k^{a}}$$

So we have $a = +2 - \eta$

On the other hand (usual chain rule for partion function)

$$\chi_T \sim G(0)$$

Assuming

$$\hat{g}(x) \to \text{Const and } x \to 0$$

then we have as

$$x = \hat{G}(0) = \ell^{-2-a} \xi^{a} + \dots$$

$$\downarrow \xi \sim t^{-\nu}$$

$$t^{-\gamma} \sim x = \ell^{-2-a}\, t^{-\nu a} = \ell^{-2-a}\, \ell^{\nu(2-\eta)}$$

$$\Rightarrow \boxed{\gamma = \nu(2-\eta)}$$

Indeed, check this is the case!

$d=2 \qquad \gamma = 7/4 \qquad \nu = 1 \qquad \eta = \frac{1}{4}$

$d=2 \qquad \gamma = 1.237 \qquad \nu = 0.630 \qquad \eta = 0.036\ldots$

Thus, from simple dimesional analysis we conclude that all the critical exponents are not independent...

We can actually conclude much more!

To do so we will make use of scaling arguments $\Rightarrow$ Emprically oberved in data by Widom + given theoretical plausibility by Kadanoff $\Downarrow$ Culminating in Wilson RG.

The general observation of Widom was that one could rescale data and show that there existed exponents c and g such that

$$\boxed{f(t\lambda^{c}, h\lambda^{g}) = \lambda f}$$

for and λ

In particular choosing $\lambda = t^{1/c}$ this implies we can write

$$f(t,h) = t^{1/c} f\left(1, \frac{h}{t^{+g/c}}\right) \equiv t^{1/c} f_t\left(\frac{h}{t^{-g/c}}\right)$$

and $\lambda = h^{1/g}$ that

$$f(t,h) = h^{1/g} f\left(\frac{t}{h^{+c/g}}, 1\right) \equiv h^{1/g} f_h\left(\frac{t}{h^{-c/g}}\right)$$

This is essentially the observation that our data should collapse..

If we measure (f,h) at t_1 and assign to a point $(f t_1^{-1/c}, h t^{g/c})$ at t_2 to get new curve, the two graphs will "collapse" on each other...

Turns out you can show this what real data does and there are only 2 independent critical exponents..

(The rest of these notes for this lecture follow Creswick Chapter 4)

We are now in position to really do a proper treatment of scaling and RG..

Our starting point will be an RG transformation

... I First we introduce new coarse grained D.O.F. defined by a length scale b

II We rescale the system (lattice constant / momentum cutoff) so that smallest scale in system is same

$$\underbrace{r' = b^{-1} r}_{\text{rescaling....}} \qquad \left(\frac{a'}{a} = b\right) \leftarrow \text{lattice spacing of momentum cutoff}$$

(So we can compare two systems)

$$\frac{\Lambda'}{\Lambda} = b$$

So notice all quantities with dimensions of length scale will like this...

In particular Volume..

$$V' = b^{-d} V$$

and free-energy density

$$f' = b^{d} f$$

Generally, one also chooses RG scheme to fix the coefficient of Gaussian term

$$H_2 = \frac{1}{k} \sum_k k^2 S(k) S(-k)$$

One of the insights of Kadanoff was

that we must also rescale these fields since in general there will "correction" to this term (bare mass vs renormalized mass)

This means $S'(k) = b^{1-\eta/2} S(k) = b^{\frac{2-\eta}{2}} S(k)$

(Recall the naive scaling (i.e dim analysis) was $\eta = 0$)

or

$$S'(x) = b^{\frac{d-2+\eta}{2}} S(x)$$

We will be especially interested in understanding behavior near a fixed point

$$R_b(H^*) = H^*$$

Linearized one has

$$\boxed{L_b \Psi_k = \lambda_k(b) \Psi_k}$$ "eigenfunctions" + "eigenvalues"

So around fixed point we can apply an RG transformation twice with scale factor b and then b' to get

$$\lambda_k(bb') = \lambda_k(b)\lambda_k(b')$$

$\Longrightarrow$

$$\lambda_k(b) = b^{y_k}$$ "y_k" is called the dimension of operator Ψ_k

and near fixed point we can write

$$H[\{h_k\}] = H^* + \sum_k h_k \Psi_k$$

Applying RG transform we find that

$$R_b(H[\{h_k\}]) = H[\{b^{y_k} h_k\}]$$

Notice if $y_k < 0$ $\quad \Psi_k$ tends to zero $\Rightarrow$ irrelevant

$y_k > 0$ $\quad \Psi_k$ grows $\Rightarrow$ relevant

$y_k = 0$ $\quad$?? $\Rightarrow$ marginal

There is always one relevant scaling field $\Rightarrow$ constant cV where V is volume

$H^* + cV$

Since V scales as b^d and Free-Energy is invariant

$$c' = b^d c \ldots$$

But we usually ignore this...

We are now in position to start doing some general scaling arguments

Recall near fixed point.

$$H[\{h_k\}] = H^* + \sum_k h_k \Psi_k$$

Then we must have after RG transformation length scales are related by b...

$$\xi(\{h_k\}) = b\,\xi(\{b^{y_k} h_k\})$$

Let us consider some special cases of this

$$\xi(\{0\}) = b\,\xi(\{0\}) \qquad \text{(at fixed point } \{\eta_K\} = 0$$

$$\Longrightarrow \underbrace{\xi = 0}_{} \quad \text{and} \quad \xi = \infty$$

Trivial fixed point (paramagnet/ordered state)

non-trivial fixed point where correlation function diverges ..

⟹ Ising like systems with single relevant variable

For simplicity let us consider a system with no external field $h=0$ and only one relevant parameter

$$t = \frac{T - T_c}{T}$$

In this case

$$\xi = b\,\xi(t b^{y_T})$$

For this to be true for all b we need

$$\xi(x) \sim x^{-\frac{1}{y_T}} \Longrightarrow$$ ~~equivalently~~

$$\Longrightarrow \quad \xi \sim |t|^{-1/y_T} \qquad \boxed{y_T = \frac{1}{\nu} = \nu^{-1}} \quad \text{Critical exponent}$$

Similarly we have

$$f(t) = b^{-d} f(t b^{y_T}) = f(t b^{\nu})$$

This holds for all b...

Lets choose $b = |t|^{-1/\nu}$

$$f(t) = |t|^{\nu d} f(\mathrm{sgn}(t)) \approx |t|$$

$\frac{t}{|t|}$

And

$$C(t) = \frac{\partial^2 f}{\partial t^2} \approx |t|^{(\nu d - 2)} f(\mathrm{sgn}(t)) \Rightarrow |t|^{-\alpha}$$

by definition

$$\Rightarrow \quad \nu d - 2 = -\alpha$$

$$\Rightarrow \quad \boxed{\alpha = 2 - \nu d}$$

Scaling relation...
exponents ~~not~~ independent..

Now consider case where $t \neq 0$ and $h \neq 0$

$$f(t,h) = b^{-d} f(t b^{y_T}, h b^{y_h})$$

$$m(t,h) = -\frac{\partial f(t,h)}{\partial h}$$

$$-\frac{\partial f(t,h)}{\partial h} = b^{-d+y_h} \frac{\partial f(t b^{y_T}, h b^{y_h})}{\partial h}$$

$$\Rightarrow \quad m' = b^{d-y_h} m \Leftarrow\Rightarrow m(t,h) = b^{-d+y_h} m(t b^{y_T}, h$$

~~$m'(t,h)$~~

Now choosing $|t| b^{y_T} = 1$ gives using $y_T = \frac{1}{\nu}$

$$|t| b^{\frac{1}{\nu}} = 1$$

$$b = |t|^{-\nu}$$

$$m(t,h) = -|t|^{\nu(d-y_h)} f_h(\mathrm{sgn}(t), h|t|^{-\nu y_h}) \quad (\star)$$

Now as $h \to 0$ we know

$$m(t, h=0) = t^{\beta}$$

So $\boxed{\beta = \nu(d - y_h)}$ (Another relationship)

Similarly taking derivative of $(\star)$ with h

$$\chi(t,h) = -|t|^{\nu(d-2y_h)} f_{hh}(\mathrm{sgn}(t), h|t|^{-\nu y_h})$$

$$\chi(t, h=0) \Rightarrow |t|^{-\gamma}$$

$$\Rightarrow \boxed{\gamma = -(d-2y_h)\nu}$$

Finally, we can return to

$$m(t,h) = b^{-d+y_h} m(t b^{y_T}, h b^{y_h}) = -b^{d+y_h} f_h(t b^{y_T}, h b^{y_h})$$

Let us now ~~write~~ choose $|h| b^{y_h} = 1$, then $b = |h|^{-\frac{1}{y_h}}$

$$m(t,h) = -|h|^{\frac{(d-y_h)}{y_h}} f_h\left(\frac{t}{h^{y_T/y_h}}, \mathrm{sgn}(h)\right)$$

So that since

$$m \sim h^{1/\delta}$$

$$\Rightarrow \boxed{\delta = \frac{y_h}{d - y_h}}$$

Finally, it will also be useful to define scaling exponent for the correlation functions...

As usual we will define generating functional

$$\mathcal{Z}(\{\eta_i\}) = \mathrm{Tr}_{\{S_i\}} \exp\left(-H[\{S_i\}] + \sum_i \eta_i S_i\right)$$

$$\Phi[\{\eta_i\}] = -\ln\left(\mathcal{Z}(\{\eta_i\})\right)$$

(note extensive so invariant under scaling)

$$G(r_{i,j}) = \frac{\partial^2 \Phi}{\partial \eta_i \partial \eta_i}(\{\eta_i = 0\})$$

Since $\sum_i \eta_i S_i$ must be invariant under scaling and S_i scales as $S_i' = b^{y_s} S_i$ we have that $\eta_i' = b^{-y_s}\eta_i$ so that

$$\Phi(t, \{\eta_i\}) = \Phi(t b^{y_T}, \{\eta_i b^{-y_s}\})$$

This implies that $\Phi[t, \{\eta_i\}]$ scales as:

$$G(r) = b^{-2y_s}\, \Phi_{\eta_i\eta_j}(t b^{y_T}, r b^{-1}, \{\eta_i = 0\})$$

Let us choose $b = r$

$$\Rightarrow \quad G(r,t) = r^{-2y_s}\, \Phi_{\eta_i\eta_j}(t r^{y_T}) \qquad y_T = \frac{1}{\nu}$$

$$= r^{-2y_s}\, \Phi_{\eta_i\eta_j}(t r^{\frac{1}{\nu}}) \qquad t = \xi^{\frac{1}{\nu}}$$

$$= r^{-2y_s}\, g\left(\frac{r}{\xi}\right) \qquad g(x) = \Phi_{\eta_i\eta_j}(x^{\frac{1}{\nu}})$$

as $t \to 0$

$$G(r,t) \to \frac{g(0)}{r^{-2y_s}}$$

$$\Rightarrow \quad \boxed{y_s = \frac{1}{2}(d - 2 + \eta)}$$

For the Ising system this is just what we are calling y_h...

Recall we have $\boxed{\alpha = 2 - \nu d}$

So we get from $\beta = \nu(d - y_h)$ that

$$\beta = \nu\left(d - \frac{1}{2}(d-2+\eta)\right)$$

$$\boxed{\beta = \frac{\nu}{2}(d-2+\eta)}$$

From $\gamma = -(d - 2y_h)\nu$ we have

$$\gamma = -(d - (d-2+\eta))\nu$$

$$\boxed{\gamma = \nu(2-\eta)}$$

From $\frac{1}{\delta} = \frac{y_h}{d - y_h}$

$$\boxed{\frac{1}{\delta} = \frac{\frac{1}{2}(d-2+\eta)}{d - \frac{1}{2}(d-2+\eta)} = \frac{d-2+\eta}{d+2-\eta}}$$

Finally, we did not derive it but can also show

$$\boxed{\alpha + 2\beta + \gamma = 2}$$

Finite Size Scaling

Before ending our discussion Scaling, we will spend a few minutes/ pages on finite size scaling...

Often in practice, (numerically especially) we are not working with infinite systems but finite size systems...

In this case ξ (corr. length) can diverge instead

$\xi \sim L$ and all singularities will be rounded off...

Because we expect $\xi \sim |t|^{-\nu}$ we expect

$$L^{-\frac{1}{\nu}} \sim T_m - T_c$$

(" shifted finite size critical temp)

So how do we make this more rigourous? The key is to treat $\frac{1}{L}$ (one over system size) as a relevant operator.

Why, we know under rescaling

$$r' = b^{-1} r$$

$$\Longrightarrow \quad \frac{1}{L'} = b \frac{1}{L}$$

(so scaling dimension 1)

From this we get scaling equation for free-energy density

$$f\left(t,\tfrac{1}{L}\right) = b^{-d} f\left(t b^{1/\nu}, b\tfrac{1}{L}\right)$$

Taking $b=L$ and differentiating twice with respect to t

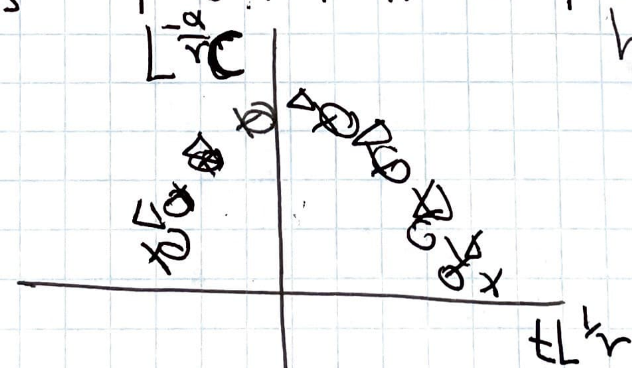
$$C\left(t,\tfrac{1}{L}\right) = b^{\frac{2}{\nu}-d} f_{tt}\left(t b^{1/\nu}, b\tfrac{1}{L}\right)$$

$$\Longrightarrow \quad C\left(t,\tfrac{1}{L}\right) = L^{\frac{2-d\nu}{\nu}} f_{tt}\left(t L^{1/\nu}, 1\right)$$

$$= L^{+\frac{\alpha}{\nu}} f_{tt}\left(t L^{1/\nu}, 1\right) \qquad \text{Use } \alpha = 2-d\nu$$

$$\Longrightarrow \quad L^{-\frac{\alpha}{\nu}} C\left(t,\tfrac{1}{L}\right) = f_{tt}\left(t L^{1/\nu}, 1\right).$$

This implies that if we plot normalized specific heat (Left side) verses $tL^{1/\nu}$ should get data collapse!!



In practice, find α and ν that maximize collapse

$\Longrightarrow$ Best estimate for α, ν from finite size data

$\Longrightarrow$ Rest of exponents follow...