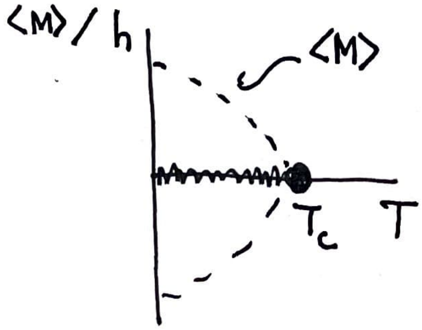
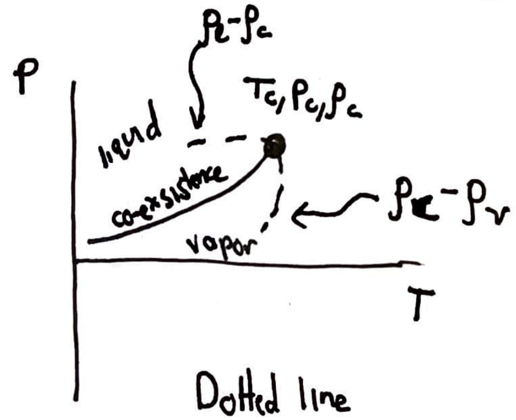
Scaling I: Landau + Critical Exponents + Non-trivial Scaling

(This first part follows Shankar Chap 12 + Goldenfeld Chapter 7) + Creswick 4.1

For then next 2-3 weeks we will now turn our attention to understanding critical phenomena and some standard advanced statistical mechanics techniques...

What was interesting is of course the theory of phase transitions... Consider two prototypical examples: Magnetic system (say Ising Model) and Liquid Vapor transition... As we know the phase diagrams for these look like





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Define a set of critical exponents in terms of "reduced temperature" $t = \dfrac{T - T_c}{T_c}$

Critical Exponents for basic Thermodynamic Functions

(1) $m = \langle M \rangle = t^{\beta}$

(2) $C_V \simeq t^{-\alpha}$

(3) $\chi \simeq \left.\dfrac{\partial \langle M \rangle}{\partial h}\right|_{h=0} \sim t^{-\gamma}$

(4) $\xi \simeq t^{-\nu}$

Scaling relations for Order Parameter m and Correlation Function at Critical point..

(5) $G(r) = \langle m(\vec{r}) m(0) \rangle \underset{(T=T_c)}{\simeq} \dfrac{1}{r^{d-2+\eta}}$

(6) $m \sim h^{1/\delta} \quad (T = T_c)$

As we discussed in our first lecture, the surprising observation was that of universality... That magnetic systems and liquid-vapor should have same <u>critical exponents</u> was mysterious... Many systems with very different <u>microscopic physics</u> give rise to the same <u>macroscopic physics</u>.

This is origin of RG in phase transitions...
In particular, a big motivation was that both <u>experiments</u> and Onsager's <u>exact solution to 2D Ising model</u> gave different <u>critical exponents</u> than Landau theory...

$\boxed{\text{Landau Theory}}$

We will now try to think about how we coarse-grain and the approximations contained in Landau construction...

The RG/Landau coarse-graining transformation ~~is~~ can be thought of as working on level of partition function...

Define coarse grained variables $\{S_i\}$ (or often magnetizations $m(\vec{r})$)

We define new Hamiltonian

$$e^{-H'[\{S_i\}]} = \mathrm{Tr}_{\{S_i\}} \underbrace{T(\{S_i\},\{S_i'\})}_{\text{Projection operator}} \overset{\text{original}}{e^{-H[\{S_i\}]}}$$

How do we construct such $T(\{S_i\},\{S_i\})$?

(A) Fundamental length scales should be related $\frac{a'}{a} = b$

(B) Coarse grained variables should have same symmetry/ dimension of internal space

(C) Homogenous states corresponding to phase should transform to similar homogenous phases

(D) Free-energy/Coarse grained partition function should be identical to the orginal free energy...

$$\underbrace{\mathop{\mathrm{Tr}}_{\{s_i\}}}_{\text{orginal variables...}} T(\{S_i\}, \{s_i\}) = 1$$

So let us consider what Landau did...

Consider Ising model with spins $\{s_i\}$ with nearest-neighbor coupling J in magnetic field H in d dimensions.

Let us "average" sites in some neighborhood with volume (b^d) $V(\vec{r})$ labels location

and define

$$T(\{m(\vec{r})\}, \{s_i\}) = \delta\Big(m(\vec{r}) - \frac{1}{[\#\text{ of Sites in } V(\vec{r})]} \sum_{i \in V(\vec{r})} s_i\Big)$$

In this case we can write the partition function as

$$Z(J,T,H) = \sum_{S_i} \exp\Big[\overbrace{\frac{1}{KT}\Big(\sum_{\langle ij\rangle} J S_i S_j + \sum_i H S_i}^{-H\{S_i\}}\Big]$$

$$= \overbrace{\sum_{S_i}}^{\mathrm{Tr}_{\{S_i\}}} \int \prod_r dm(\vec r)\, \delta\Big(m(\vec r) - \frac{1}{\#\text{of site is } \mathcal{V}(\vec r)} \sum_{i\in \mathcal{V}(\vec r)} S_i\Big)\, e^{-\frac{H\{S_i\}}{K_B T}}$$

$$\equiv \prod_r \int dm(\vec r)\, e^{-f[m(\vec r)]}$$ } definition of Landau free energy (functional) (Sufficiently "smoothened") (really Boltzmann Weight!)

$$\equiv \int [Dm]\, e^{-f[m(\vec r)]}$$

So now we have

free energy per unit weight

$$Z(t,h) = e^{-f(t,h)} = \int [Dm]\, e^{-f[m(r)]} \approx e^{-f}$$

The Landau approximation is

$$f(t,h) \simeq f(m^*)$$

We have replaced this by minimum value...?

Ignored all fluctuations!!

when m^* is minimum of $f[m(r)]$

(Note in Shankar the factors of volume are quite confusing!!)

Now we invoke usual magic of Landau

- $f(m)$ is analytic can be expanded in power series
- $h=0$, $f(m)$ is even
- Coefficient will be analytic in T
- When correlations are large, only long-wavelength / low derivatives will contribute ..

For $h=0$

$$f(m) = r_0(T)\,m^2 + u_0(T)\,m^4$$

$$0 = \left.\frac{df}{dm}\right|_{m^*} = 2r_0(T)\,m_* + 4u_0(T)\,m_*^3$$

$\Rightarrow$

$r_0(T) > 0$ $\qquad m^* = 0$

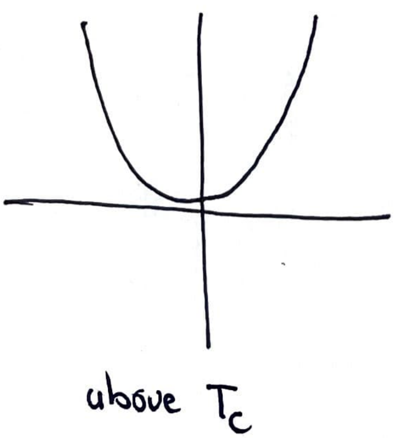
If

$r_0(T) < 0$ $\qquad m^* = 0$ $\qquad$ (local max)

$$m^* = \pm\sqrt{\frac{2|r_0(T)|}{4u_0(T)}}$$

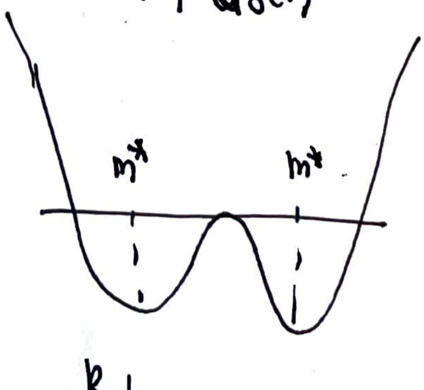
(Symmetry breaking solution)

f(m)



above T_c

$r_0(T) > 0$



Below T_c

$r_0(T) < 0$

So we know transition corresponds to $r(T) = 0$

So can expand

$$r_0(T) = a(T - T_c) \simeq t$$

So we have that

$$m^*(t) \approx |t|^{1/2} \Longrightarrow \boxed{\beta = 1/2}$$

Landau critical exponent..

$$C_V = \frac{d^2 f}{dt^2} \Rightarrow \frac{d^2}{dt^2}\left[r_0(t)(m^*)^2 + u_0(t)(m^*)^4\right]$$

leading term $t \to 0$

$$\Rightarrow \frac{d^2}{dt^2} t^2 \approx \text{Const}$$

for $t < t_c$ and 0 for $t > t_c$

$$C_V \Rightarrow \text{constant}, \quad T < T_c$$
$$= 0 \quad T > T_c$$

$\boxed{\alpha \text{ jumps discontinuously}}$

Now consider $h \neq 0$ $r_0(T) = 0$ (at critical point)

$$f(m^*) = u_0(T_c) m^4 - hm$$

$$0 = \left.\frac{df}{dm}\right|_{m^*} = 4 u_0(T_c) m^{*3} - h$$

$$\Rightarrow m^* \propto h^{1/3} \approx h^{1/\delta} \qquad \boxed{\delta = 3}$$

We need two more critical exponents for correlation length ξ and green's function $G(\vec{r})$

We will make use of the fact that

$$G(\vec{r}) = \langle m(r)\, m(0) \rangle \sim \int \frac{d^dk}{(2\pi)^d} \frac{e^{ik\cdot r}}{k^2 + r_0}$$

for a system describe by

$$f = \int d^d r \left[\frac{1}{2}(\nabla m)^2 + \frac{1}{2} r_0(T) m^2\right] \quad (\text{H.W.})$$

$$Z(T) = \int [\mathcal{D}m]\, e^{-[f(m,T)]}$$

$$= \int [\mathcal{D}m]\, e^{-\int d^d r \left[\frac{1}{2}(\nabla m)^2 + \frac{1}{2} r_0(T) m^2\right]}$$

$$= \int [\mathcal{D}m]\, e^{-\int d^d r\, m(\vec{r}) [\nabla^2 + r_0(T)] m(\vec{r})}$$

Green's function is inverse of this operator

$$\underset{\substack{r_0 = 0 \\ T = T_c}}{G(r)} = \int \frac{d^d k}{(2\pi)^d} \frac{e^{ikr}}{k^2} \simeq \frac{1}{r^{d-2}} \qquad \boxed{\eta = 0}$$

Compare $\underset{T=T_c}{G(r)} \sim \frac{1}{r^{d-2+\eta}}$

When $T > T_c$ one can do integral (Contour)

$$G(r) \sim \frac{1}{4\pi r} e^{-r\sqrt{r_0}}$$

$$\xi \sim \frac{1}{\sqrt{r_0}} \sim \frac{1}{\sqrt{t}} \sim \frac{1}{t^{\nu}} \qquad \boxed{\nu = \frac{1}{2}}$$

So we have Landau exponents

	Landau	d=2 (Ising	d=4
α	Jump	0^+	0^+
β	$\frac{1}{2}$	$1/8$	$1/2$
γ	1	$7/4$	1
δ	3	15	3
ν	$1/2$	1	$1/2$
η	0	$1/4$	0

So Landau is exact in d=4 and completely fails in d=2

Before resolving this "puzzle" (Failure of Landau), lets add to the mystery...

Actually, there is another way to analyze Landau theory using just dimensional analysis

Let us start with

$$Z = \int Dm \, e^{-\int d^d r \left[\frac{1}{2}(\nabla m)^2 + \frac{1}{2} r_0 m^2 + \frac{1}{4} v_0 m^4 \right]}$$

Recall $r_0(t) \sim t \qquad t \approx \frac{T-T_c}{T_c}$

So now look at dimensions

$$\left[\int d^d r \, (\nabla m)^2 \right] \sim 1 \Rightarrow L^d \cdot L^{-2} [m]^2 = 1$$

$$[m] = L^{1-d/2}$$

$$\left[\int d^d r \, r_0 m^2 \right] = 1 \Rightarrow L^d [r_0] L^{2-d}$$

$$[r_0] = L^{-2}$$

$$\left[\int d^d r \, v_0 m^4 \right] = 1 \Rightarrow L^d [v_0] L^{4-2d} = 1$$

$$[v_0] = L^{d-4}$$

So now we can think about how things scale

$$\xi \sim [L]$$

$$\Rightarrow \xi \sim r_0^{-1/2} f(\ )$$

dimensionless parameters

(ca

came back to this

$$\sim t^{-1/2} \Rightarrow \boxed{\nu = 1/2}$$

Simply dimensional analysis

$$G(\vec{r}) = \langle m(\vec{r})\, m(0) \rangle \sim L^{2-d}$$

Useful to just work in Fourier space

So if we rescale $L \to bL$

$r \to r' = br$

$$\hat{G}(k) = \int d^d r\, e^{-ikr} G(r)$$

$$[\hat{G}(k)] \sim L^{2-d} \cdot L^d$$

$$[\hat{G}(k)] \sim L^2$$

$$\hat{G}(k') = b^{-2} \hat{G}(k)$$

But we know by assertion

$$\underset{\substack{t=0 \\ T=T_c}}{G(r)} \sim \frac{1}{r^{d-2+\eta}} \Rightarrow [G(r)] \sim L^{2-d-\eta}$$

$$\underset{\substack{T=T_c \\ t=0}}{\hat{G}(k)} \sim k^{-2+\eta}$$

Both imply

$$\boxed{\eta = 0}$$

Another Landau exponent..

But we know that for $d<4$, Landau and hence these scaling argument are wrong!!

How could this be???

Well we know the answer actually is that there is another length scale in the problem we assummed did not matter at long lengths

$\Rightarrow$ the lattic spacing a.

So let us return

$$[\xi] = L \quad [a] = L \quad [r_0] \sim L^{-2}$$

$$\xi = r_0^{-1/2} f(r_0 a^2)$$ dimensionless scale

as $t \to 0$ $r \to 0$ so need to know what $\lim_{x \to 0} f(x)$ ... If it is constant we recover the old result but if

$$f(x) \sim x^{\theta} \quad \text{as } x \to 0$$

θ is the "anamolous dimension"

$$\xi \sim r_0^{-1/2} r_0^{\theta} a^{2\theta} \sim t^{-\frac{1}{2}+\theta} a^{2\theta}$$

$$\nu = \frac{1}{2} - \theta$$

Thus, we see that in both pictures, the break down of Landau scaling comes from ignoring small scale physics...

Manifests as

1. Taking saddle point in coarse graining and ignoring fluctuations
2. Assuming we can ignore <u>microscopic length scales</u>

Finally, before moving on to how Widom, Kadanoff, and ultimately Wilson fixed this let us show that dimensional analysis also can tell us when this theory fails...

Let us return to

$$Z = \int Dm\, e^{-\int d^d r \left(\underbrace{\left[\frac{1}{2}(\nabla m)^2 + \frac{1}{2} r_0 m^2(\vec{r})\right]}_{H_0 \text{ (non-interacting)}} + \underbrace{\frac{U_0}{4} m^4(\vec{r})}_{\text{treat as interaction}} \right)}$$

$$= \int \mathcal{D} m\, e^{-\int d^d r\, H_0} \left[1 - \int \frac{d^d r}{4} U_0 m^4(\vec{r}) + \frac{1}{2!}\left(\int \frac{d^d r}{4} U_0 m^4(\vec{r})\right)^2 + \dots \right]$$

Let us actually rescale this to make coupling we are doing perturbation in dimensionless.

$\vec{r} \sim [L]^{-2}$ $[m] \sim L^{1-d/2}$ $u \sim L^{d-4}$

$$\vec{x} \equiv \vec{r}\sqrt{r_0} \qquad \phi = \frac{m(\vec{r})}{r_0^{\frac{d-2}{4}}} \qquad \bar{u}_0 \sim \frac{u_0}{r_0^{-\frac{d}{2}+2}} \sim u_0 r_0^{\frac{d-4}{2}}$$

In terms of these variables

$$Z = \int \mathcal{D}\varphi \, e^{\int d^dx \left[\frac{1}{2}\nabla^2\phi(x) + \frac{1}{2}\phi^2(x)\right] + \frac{1}{4}\bar{u}_0 \phi^4(x)]}$$

But now doing perturbation theory in

$$\bar{u}_0 \sim u_0 r_0^{\frac{d-4}{2}}$$

As $t \to 0$ $r_0 \to 0$ so for $d<4$ $\bar{u}_0 \to \infty$
This means that we know Landau should fail in $d<4$!!